Problem Set 9

Due: Friday, March 23

Problem 1. Let f^λ be the number of SYT of shape λ. Find (and prove) a simple formula for the generating functions $\sum_{n\geq 0} a_n \frac{x^n}{n!}$ and $\sum_{n\geq 0} b_n \frac{x^n}{n!}$, where

$$a_n = \sum_{\lambda \vdash n} (f^\lambda)^2,$$

$$b_n = \sum_{\lambda \vdash n} f^\lambda.$$

Problem 2. Prove that the set $\{r_l(p_\lambda) : \ell(\lambda') \leq l\}$ is a basis for Λ_l, the ring of symmetric functions in the variables $x_1, x_2, \ldots, x_l$.

Problem 3. Express $r_2(h_n)$ in terms of the basis $\{r_2(h_\lambda) : \ell(\lambda') \leq 2\}$ of Λ_2.

Problem 4. An *increasing subsequence* of a permutation $\pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ is a subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_j}$ such that $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_j}$. For instance, 2367 is an increasing subsequence of the permutation 52386417. Show that for any $\pi \in \mathcal{S}_n$, $\mathrm{sh}(P(\pi))_1$ is the length of the longest increasing subsequence of π, where $\mathrm{sh}(P(\pi))$ is the shape of the insertion tableau of π.

Problem 5. Fix a partition μ of $n-1$. Find (with proof) a simple formula for $\sum_\lambda f^\lambda$ in terms of f^μ, where the sum is over all partitions of n obtained by adding a square to the diagram of μ.