Problem Set 6

Due: Friday, February 17

Problem 1. An arrangement $\mathcal{A}$ is in *general position* if for any $\{H_1, \ldots, H_p\} \subseteq \mathcal{A}$, there holds

$$\begin{cases} \dim(H_1 \cap \cdots \cap H_p) = n - p & \text{if } p \leq n, \\ H_1 \cap \cdots \cap H_p = \emptyset & \text{if } p > n. \end{cases}$$

Let $\mathcal{A}_m^3$ be the arrangement in $\mathbb{R}^3$ consisting of m planes in general position. Determine (with proof) the number of regions $r(\mathcal{A}_m^3)$ of $\mathcal{A}_m^3$ without using Theorem 2.5.

Problem 2. Let G be a forest (graph with no cycles) on the vertex set $[n]$. Show that $L_G \cong B_{E(G)}$, the boolean lattice of all subsets of $E(G)$. Here, $L_G \cong L(\mathcal{A}_G)$ is the lattice of contractions of G.

Problem 3. Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$. Suppose that $\chi_{\mathcal{A}}(t)$ is divisible by t^k but not t^{k+1}. Show that $\text{rank}(\mathcal{A}) = n - k$.

Problem 4. Let $\mathcal{A}$ be the arrangement in $\mathbb{R}^n$ consisting of the n hyperplanes

$$x_1 = x_2, x_2 = x_3, \ldots, x_{n-1} = x_n, x_n = x_1.$$

Compute (with proof) the characteristic polynomial $\chi_{\mathcal{A}}(t)$ and the number $r(\mathcal{A})$ of regions of $\mathcal{A}$.

Problem 5. A *face* of an arrangement $\mathcal{A}$ is a set $\emptyset \neq F = \bar{R} \cap x$, where $R \in \mathcal{R}(\mathcal{A})$ and $x \in L(\mathcal{A})$ and $\bar{R}$ is the closure of R. Let $f(n)$ be the total number of faces of the braid arrangement $\mathcal{B}_n$. Find (with proof) a simple formula for the generating function

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = 1 + x + 3\frac{x^2}{2} + 13\frac{x^3}{3!} + 75\frac{x^4}{4!} + 541\frac{x^5}{5!} + 4683\frac{x^6}{6!} + \cdots .$$