Problem Set 5

Due: Friday, February 10

Problem 1. Prove that the q-multinomial coefficients satisfy the recurrence

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_r \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k_1 - 1, k_2, \ldots, k_r \end{bmatrix}_q + q^{k_1} \begin{bmatrix} n-1 \\ k_1, k_2 - 1, \ldots, k_r \end{bmatrix}_q + \cdots + q^{k_1 + \cdots + k_{r-1}} \begin{bmatrix} n-1 \\ k_1, k_2, \ldots, k_r - 1 \end{bmatrix}_q .$$

Problem 2. A polynomial $f(q) = c_0 + c_1 q + \cdots + c_n q^n$ is *symmetric* if $c_i = c_{n-i}$ for all i. The polynomial $f(q)$ is *unimodal* if there exists a j such that $c_0 \leq c_1 \leq \cdots \leq c_j \geq c_{j+1} \geq \cdots \geq c_n$. Prove that for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$, the *$q$-multinomial coefficient*

$$\begin{bmatrix} k_1 + \cdots + k_r \\ k_1, \ldots, k_r \end{bmatrix}_q = \frac{[k_1 + \cdots + k_r]!}{[k_1]! \cdots [k_r]!}$$

is a symmetric and unimodal polynomial with nonnegative integer coefficients. (You can use the fact that the q-binomial coefficients are symmetric unimodal polynomials with nonnegative integer coefficients.)

Problem 3. Determine the number of permutations of $[n]$ with an odd number of cycles all of which have odd lengths. For example, for $n = 3$, there are 3 such permutations: $(1)(2)(3), (123), (132)$.

Problem 4. Prove the following q-analog of the convolution formula for binomial coefficients.

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_q = \sum_{i+j=k} q^{(m-i)j} \begin{bmatrix} m \\ i \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q$$

Problem 5. Prove that the number of ways to write the n-cycle permutation $(123 \cdots n)$ as a product of $n-1$ transpositions is equal to n^{n-2}. Here, product refers to multiplication in the group $\mathcal{S}_n$. A transposition is the same as a 2-cycle.