# Problem Set 4

Due: Friday, February 3

**Problem 1.** Find a simple expression for the ordinary generating function in two variables

$$F(x,y) = \sum_{n,k\geq 0} \binom{n}{k} x^n y^k.$$

**Problem 2.** A *perfect matching* on a set $S$ of $2n$ elements is a set partition of $S$ into $n$ blocks of 2 elements each. Taking $S = [2n] = \{1, 2, \ldots, 2n\}$, and thinking of the blocks in a matching as the edges of a graph, call edges of the form $\{i, i+1\}$ *short*, and all other edges *long*. Let $M_n(x)$ be the ordinary generating function that counts perfect matchings on $[2n]$ with weight $x^s$, where $s$ is the number of short edges, so for instance $M_2(x) = 1 + x + x^2$. Prove the recurrence

$$M_n(x) = (x + 2n - 2)M_{n-1}(x) + (1 - x)\frac{d}{dx}M_{n-1}(x).$$

**Problem 3.** Suppose that $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are sequences such that $b_n = \sum_{k=1}^{n} S(n,k)a_k$; therefore, since $s$ is the inverse of $S$, $a_n = \sum_{k=1}^{n} s(n,k)b_k$. Prove that

$$B(x) = A(e^x - 1),$$

where $A(x) = \sum_{n\geq 1} a_n \frac{x^n}{n!}$ and $B(x) = \sum_{n\geq 1} b_n \frac{x^n}{n!}$.

**Problem 4.** Prove that

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k}.$$

Here, $(x+y)_n$ is the falling factorial $(x+y)(x+y-1)\cdots(x+y-n+1)$.

**Problem 5.** Show that $C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$ is a polynomial in $q$ with nonnegative integer coefficients. This is a $q$-analogue of the Catalan numbers.