Problem Set 3

Due: Friday, January 27

Problem 1. Consider the 64 possible sums of the form

$$\epsilon_1 + \epsilon_2 + 2\epsilon_3 + 5\epsilon_4 + 10\epsilon_5 + 10\epsilon_6,$$

where each ϵ_i is 0 or 1. Let a_n be the number of different sums that represent n. Give a simple expression for the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.

Problem 2. Let π be a random permutation of $1, 2, \ldots, n$ (chosen from the uniform distribution). Fix a positive integer $1 \leq k \leq n$. What is the probability that in the cycle notation of π, the length of the cycle containing 1 is k?

Problem 3. Let F_n be the n-th Fibonacci number. Prove or disprove: the generating function $G(x) = \sum_{n\geq 0} F_n^2 x^n$ is rational.

Problem 4. Let k and n be integers with $0 \leq k < n$. Evaluate the sum $\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^k$ (and prove that your answer is correct).

(Suggestion: begin by making a table of the answer for small k and n.)

Problem 5. Let f_n be the number of permutations of $[n]$ with no fixed points or cycles of size 2. Show that the exponential generating function $F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ is equal to

$$\frac{e^{-x-\frac{x^2}{2}}}{1-x}.$$