Problem Set 11

Due: Friday, April 6

Problem 1. For $\lambda \vdash n$, $\ell(\lambda) \leq 2$, express h_λ and e_λ in terms of the basis $\{s_\lambda : \lambda \vdash n\}$ of $\Lambda^n_{\mathbb{Q}}$. Use the expansion of $h_{\lceil n/2 \rceil} h_{\lfloor n/2 \rfloor}$ to give a short proof of the fact that the number of SYT of size n and at most 2 rows is $\binom{n}{\lfloor n/2 \rfloor}$ (problem 2, problem set 8).

Problem 2. Let $\delta^n = (n-1, n-2, \ldots, 1)$ be the staircase shape. Let $\nu^{r,c}$ denote the rectangle shape (c^r) with c columns and r rows. Express the skew Schur functions $s_{\delta^{n+1}/\delta^n}$ and $s_{\nu^{r,c}/\nu^{r-1,c-1}}$ in terms of Schur functions.

Problem 3. Determine $\sum_{\pi \in \mathcal{S}_n} p_{\rho(\pi)}$ in terms of the basis $\{h_\lambda : \lambda \vdash n\}$ of $\Lambda^n_{\mathbb{Q}}$. Here p denotes the power sum symmetric function and $\rho(\pi)$ is the cycle type of π.

Problem 4. Verify the identity

$$\prod_i (1 - q x_i)^{-1} \cdot \prod_{i<j} (1 - x_i x_j)^{-1} = \sum_\lambda q^{c(\lambda)} s_\lambda(x),$$

where $c(\lambda)$ denotes the number of parts of λ' that are odd. (Suggestion: Generalize the proof of Corollary 7.13.8 of Stanley. If you use this proof, be sure to fill in the details not given in the book.)

Problem 5. Let $\Psi : \Lambda \to \mathbb{Q}[t]$ be the specialization defined by $\Psi(p_n) = 1 - (-t)^n$, $n > 0$. Show that

$$\Psi(s_\lambda) = \begin{cases} t^k(1+t) & \lambda = \langle n-k, 1^k \rangle,\ 0 \leq k \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$