Problem Set 10

Due: Friday, March 30

Problem 1. Draw the growth diagram for the permutation 6127453. Identify which of the local rules L1–L4 is used at each position.

Problem 2. Let $U : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ and $D : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ be linear transformations defined by $U(f) = p_1 f$ and

$$D(f) = \frac{\partial}{\partial p_1} f,$$

where $\partial/\partial p_1$ is applied to f written as a polynomial in the p_i's.

a. Show that $DU - UD = I$, the identity operator.

b. Show that $DU^k = kU^{k-1} + U^k D$.

Problem 3. Let $\lambda \vdash n$. Show that $\left(\prod_i (m_i(\lambda)!)^{-1}\right)\langle p_1^n, h_\lambda\rangle$ is equal to the number of set partitions of $[n]$ with block sizes $\lambda_1, \lambda_2, \ldots$.

Problem 4. Let $\mathbf{k} \in \{1,2\}^n$ be a word of length n in the alphabet $\{1,2\}$. The *diagram* of $\mathbf{k}$ is obtained from $\mathbf{k}$ by thinking of 2s as left parentheses and 1s as right parentheses and then drawing an arc between matching parentheses pairs. Express $P(\mathbf{k})$ and $Q(\mathbf{k})$ in a simple way in terms of the diagram of $\mathbf{k}$ (it may be helpful to have the following notation for the diagram of $\mathbf{k}$: let $l_1 < \cdots < l_j$ be the positions of the left-hand ends of the arcs, let $r_1 < \cdots < r_j$ be the positions of the right-hand ends of the arcs, and let u_1 and u_2 be the number of unmatched 1s and 2s respectively).

Problem 5. In how many ways can we begin with the empty partition $\emptyset$, then add $2n$ squares one at a time (always keeping a partition), then remove n squares one at a time, then add n squares one at a time, and finally remove $2n$ squares one at a time, ending up at $\emptyset$?