Problem Set 1

Due: Friday, January 13

Problem 1. A *fixed point free involution* in $\mathcal{S}_{2n}$ is a permutation $\pi \in \mathcal{S}_{2n}$ satisfying $\pi^2 = \mathrm{id}$ and $\pi(i) \neq i$ for all $i \in [2n]$. Prove that the number of fixed point free involutions in $\mathcal{S}_{2n}$ is $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

Problem 2. Determine the number of compositions of n (with any number of parts). Give a direct combinatorial proof that this value is correct.

Problem 3. Let $p(c, r, n)$ be the number of partitions of n with length at most r and largest part $\leq c$ (this is the number of partitions whose Ferrers diagram is contained in an $r \times c$ rectangle). Determine $\sum_{n \geq 0} p(c, r, n)$ and give a direct combinatorial proof that this value is correct.

Problem 4. Given $\pi \in \mathcal{S}_n$, let $c_1(\pi)$ be the number of fixed points of π (i.e. the number of cycles of π of length 1). Prove that $\sum_{\pi \in \mathcal{S}_n} c_1(\pi) = n!$.

Problem 5. For $n \geq 2$, prove that the number of compositions of n with an even number of even parts equals the number of compositions of n with an odd number of even parts.