Take home midterm

Due: Friday, February 24 at the beginning of class

Rules: You may consult your notes and the textbooks Vol. 1 and 2 of Stanley, Generating Functionology by Wilf, and Stanley's notes on hyperplane arrangements; you may **not** consult with sources other than these or other people.

Problem 1. Let a_n be the number of subsets of $[n]$ that contain no two consecutive elements. Determine the sequence $a_0, a_1, a_2, \ldots$, which begins $a_0 = 1, a_1 = 2, a_2 = 3$.

Problem 2. Let T_n denote the number of set partitions of $[n]$ with an even number of blocks, all of which have even size. For example, $T_4 = 3$, corresponding to the set partitions $12|34, 13|24, 14|23$ of $[4]$. Determine the exponential generating function $T(x) := \sum_{n\geq 0} T_n \frac{x^n}{n!}$.

Problem 3. Let P_n denote the poset of set partitions of $[n]$ ordered by refinement: $\{B_1, B_2, \ldots, B_k\} \leq \{B'_1, B'_2, \ldots, B'_{k'}\}$ if each B_i is contained in some B'_j. Let $\hat{0} \in P_n$ be the set partition with n blocks of size 1 and $\hat{1} \in P_n$ be the set partition with 1 block of size n. Determine $\mu(\hat{0}, \hat{1})$, where μ is the Möbius function of P_n.

Problem 4. Recall that the Eulerian number $A(n, k)$ is the number of permutations of $[n]$ with $k-1$ descents. There are $n!$ sequences of integers $(b_1, \ldots, b_n)$ such that $0 \leq b_i \leq n-i$ for all i. Let $B(n, k)$ be the number of these sequences $(b_1, \ldots, b_n)$ such that $|\{b_1, \ldots, b_n\}| = k$. Prove that $B(n, k) = A(n, k)$.

Problem 5. Prove that

$$\prod_{i=1}^{s}(1 + x^{-1}q^i) \prod_{i=0}^{t-1}(1 + xq^i) = \sum_{j=-s}^{t} q^{\binom{j}{2}} \begin{bmatrix} s+t \\ s+j \end{bmatrix}_q x^j.$$

Problem 6. Let $\mathcal{A}$ be an arrangement in the n-dimensional vector space V whose normals span a subspace W, and let $\mathcal{B}$ be another arrangement in V whose normals span a subspace Y. Suppose that $W \cap Y = \{0\}$. Show that

$$\chi_{\mathcal{A}\cup\mathcal{B}}(t) = t^{-n} \chi_{\mathcal{A}}(t) \chi_{\mathcal{B}}(t).$$

Problem 7. Evaluate the sum below (and prove that your answer is correct):

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{2k}{k} \binom{2n-2k}{n-k}.$$

Problem 8. A *prime parking function* is a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ of positive integers that contains at least $k+1$ entries $\leq k$, for $k = 1, \ldots, n-1$. Prove that the number of prime parking functions is $(n-1)^{n-1}$.