Homework Set 9

Due March 27

Section 5.1
Problems 14, 20, 26, 32.
Section 5.2
Problems 2, 16, 19, 20.
Section 5.3
Problems 12, 22, 24, 28.

The Proof Problems:

PROBLEM 9.1. Let A be both diagonalizable and invertible. Show that A^{-1} is diagonalizable. What is the connection between the eigenvalues of A^{-1} and those of A?

PROBLEM 9.2. Let A and B be two $n \times n$ matrices such that $AB = BA$, and assume that A has n distinct eigenvalues.

(a) If λ is an eigenvalue of A, prove that $\dim \mathrm{Nul}(A - \lambda I) = 1$.

(b) Prove that every eigenvector of A is also an eigenvector of B.

PROBLEM 9.3. For a polynomial $p(x)$ and an $n \times n$ matrix A, let $p(A)$ denote the matrix obtained by "plugging in" A for x. For example, if $p(x) = x^3 - 2x^2 + 3$, then $p(A) = A^3 - 2A^2 + 3I$.

(a) If λ is an eigenvalue of an $n \times n$ matrix A, prove that $p(\lambda)$ is an eigenvalue of $p(A)$. (Hint: use the same eigenvectors that A has.)

(b) If P is invertible, show that the following equality of matrices holds: $p(P^{-1}AP) = P^{-1}p(A)P$.

(c) If A is diagonalizable, prove that all eigenvalues of $p(A)$ are of the form $p(\lambda)$ for some eigenvalue λ of A.
(*Note*: This statement is true even if A is not diagonalizable!)

PROBLEM 9.4. Let A be a 2×2-matrix such that $A^2 = I$. Prove that A is diagonalizable. (Hint: show that either $A + I$ is not invertible or $A - I = 0$.)