Homework Set 8

Due March 20

NOTE: Please hand in the book and proof problems separately.

Section 4.7
Problems 6, 10, 12, 14, 16
Section 5.4
Problems 6, 10, 20, 24, 28

The Proof Problems:

PROBLEM 8.1: Let V be a finite dimensional vector space of dimension n with basis $\mathcal{B}$. Let $\mathcal{L}(V)$ be the vector space of linear transformations $T : V \to V$, and if $T \in \mathcal{L}(V)$, then let $[T]_{\mathcal{B}}$ denote the matrix of T relative to the basis $\mathcal{B}$.

a. Prove that the function $[\cdot]_{\mathcal{B}} : \mathcal{L}(V) \to M_{n\times n}(\mathbb{R})$, given by $T \mapsto [T]_{\mathcal{B}}$, is a (vector space) isomorphism.

b. Let $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if $[T]_{\mathcal{B}}$ is invertible.

c. Let $T \in \mathcal{L}(V)$. Prove that $\dim(\ker T) = \dim(\mathrm{Nul}([T]_{\mathcal{B}}))$.

PROBLEM 8.2: Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{v}\}$ be a basis of $\mathbb{R}^4$, where the $\mathbf{e}_i$ are the standard basis vectors.

a. Find a vector $\mathbf{v}$ such that

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Prove that your answer is correct.

b. Prove that the first three columns of $[T]_{\mathcal{B}}$ do not depend on $\mathbf{v}$.

c. Prove that there is no choice of $\mathbf{v}$ such that

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

PROBLEM 8.3: Let $T : V \to W$ be a linear transformation from the vector space V to the vector space W. Let $\mathcal{B}$ be a basis for V and let $\mathcal{C}$ be a basis for W. Let $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ be the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$ (as defined in equation (4) on page 289). Let r be the rank of $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$.

a. Prove that there exists a basis $\mathcal{D}$ of W such that $[T]_{\mathcal{D}\leftarrow\mathcal{B}}$ has exactly r nonzero rows.

b. Prove that $r = \dim(\text{range}(T))$.

c. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$ and suppose

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Prove that there does not exist a basis $\mathcal{A}$ of V such that $[T]_{\mathcal{C}\leftarrow\mathcal{A}}$ has exactly two nonzero rows.