Homework Set 6

Due February 27

NOTE: Please hand in the book and proof problems separately.

Section 4.1
Problems 7, 8, 22, 32
Section 4.2
Problems 6, 15, 24, 33
Section 4.3
Problems 8, 11, 26, 34

The Proof Problems:

PROBLEM 6.1: Let V, W, U be vector spaces and let $T : V \to W$ and $S : W \to U$ be linear transformations. Prove that the range of T is contained in the kernel of S if and only if the kernel of $S \circ T$ is equal to V.

PROBLEM 6.2: Let $\mathbf{v}$ be a fixed vector in $\mathbb{R}^n$ and let S be the set of $m \times n$ matrices A such that $A\mathbf{v} = \mathbf{0}$.

(a) Prove that S is a subspace of $M_{m\times n}$.

(b) In the special case where $n = m = 2$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, find a basis for S.

PROBLEM 6.3: Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Consider $f(t) = e^{rt}$ and $g(t) = e^{st}$ in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ for real numbers s and r. Prove that $\{f, g\}$ is linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ if and only if $r \neq s$.

PROBLEM 6.4: Let $p(x)$ be a polynomial in $\mathbb{P}_n$.

(a) Prove that, if $p(x)$ has degree d and k is an integer such that $0 \leq k \leq d$, then the kth derivative $p^{(k)}(x)$ has degree $d - k$.

(b) Show that the set of derivatives

$$S = \left\{p(x), p'(x), p''(x), \ldots, p^{(n)}(x)\right\}$$

is linearly independent if and only if $p(x)$ has degree n.