Homework Set 11

Due April 17

NOTE: Please hand in the book and proof problems separately.

Section 6.3
Problems 4, 8, 14, 21
Section 6.4
Problems 6, 10, 14, 22
Section 6.5
Problems 4, 12, 18a, 18b, 18c, 18d.

The Proof Problems:

PROBLEM 11.1:
Let $W \subset \mathbb{R}^n$ be a subspace.

a) Note that, in the book, Theorem 11 is proved using Theorem 8 and Theorem 8 is proved using Theorem 11. Prove Theorem 11 without citing Theorem 8. (Hint: verify $\{\overrightarrow{v_1}, ..., \overrightarrow{v_p}\}$ is an orthogonal set directly.)

b) Show that $W \cap W^\perp = \{\overrightarrow{0}\}$.

c) Show that $\dim(W)+\dim(W^\perp) = n$.

PROBLEM 11.2:
Let $W \subset \mathbb{R}^n$ be a subspace, and define

$$\begin{aligned} T:\ \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \overrightarrow{x} &\longmapsto \mathrm{Proj}_W \overrightarrow{x}, \end{aligned}$$

where $\mathrm{Proj}_W \overrightarrow{x}$ is the orthogonal projection of $\overrightarrow{x}$ onto W. Let A be the associated standard matrix (by now you should have already proven that T is linear in the book problems).

a) Find $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

b) Using the geometry of the problem, find as many linearly independent eigenvectors of A as you can, and describe the associated eigenvalues. (Hint: think geometrically.)

c) Show that A is diagonalizable.

d) Let $\mathcal{B} = \{\overrightarrow{w_1}, \overrightarrow{w_2}, ..., \overrightarrow{w_p}\}$ be a fixed basis of W, and $\mathcal{B}^\perp = \{\overrightarrow{w_{p+1}}, \overrightarrow{w_{p+2}}, ..., \overrightarrow{w_n}\}$ a fixed basis of $W^\perp$. Using b) and c), find an expression for A in terms of $\overrightarrow{w_1}, \overrightarrow{w_2}, ..., \overrightarrow{w_n}$.

PROBLEM 11.3:

a) Show that the matrices

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

are orthogonal.

b) Show that if U is an $n \times n$ orthogonal matrix, then $\det(U) = \pm 1$.

c) Show that if $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 orthogonal matrix, then there exists $\theta \in [0, 2\pi)$ such that U is in either of the following forms:

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \text{or} \quad U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

PROBLEM 11.4 (will not be graded):

Define the vector space of functions

$$X := \{f : [0,1] \to \mathbb{C} \big| f(t) \text{ is continuous}\}.$$

(a complex function $f(t)$ is continuous if both $\Re(f(t))$ and $\Im(f(t))$ are continuous). Define also

$$< f(t), g(t) > := \int_0^1 f(t)\overline{g(t)}dt.$$

Finally, for $n \in \mathbb{Z}$ we define $u_n(t) := e^{2\pi i n t}$.

a) Show that if $n \neq m$, then $< u_n(t), u_m(t) >= 0$. (You can use the usual calculus rules, such as $\int_a^b e^{zt}dt = (e^{zb} - e^{za})/z$.)

b) Show that $< u_n(t), u_n(t) >= 1$.

c) For $n \in \mathbb{Z}$, compute $a_n :=< t, u_n(t) >$.

d) Using a computer, make a graph of the real part of

$$\sum_{n=-N}^{N} a_n e^{2\pi i n t},$$

with various large values of N. Interpret your results by making an analogy with Theorems 5 and 8.