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Adhesive Contact During the Oblique Impact of Elastic Spheres

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1. Introduction

In a previous paper [1], a numerical solution has been developed for the quasi-static problem of the oblique impact of elastic spheres of similar materials.

Three distinct contact regimes can be identified:

(i) *Gross slip* in which tangential relative motion (slip) occurs throughout the contact area;
(ii) *Microslip* in which part of the contact area (usually a central circle) is in adhesive contact, whilst the remainder slips;
(iii) *Adhesion* in which there is no relative motion at any point in the contact area.

In the numerical solution, it was found that, for low angles of incidence, adhesion occurred during most of the first part of the cycle, within the limits of accuracy of the numerical procedure. A similar conclusion was reached by Mindlin and Deresiewicz [2] for the related problem of the constant direction oblique indentation of a half space by a sphere. For higher angles of incidence, the impact started with a period of gross slip which appeared to give way instantaneously to complete adhesion in certain cases.

In this paper, the problem will be investigated in a more rigorous manner, using an analytic representation of the traction and displacement distributions. A differential equation will be derived and solved describing the adhesive phase of the impact.

It is assumed throughout that there is a constant limiting ratio between tangential and normal surface traction—the coefficient of friction—which cannot be exceeded and which is achieved in all slip regions.

For simplicity, we restrict attention to the case of a sphere impacting a half space, since the more general problem presents no new features. The two solids are assumed to be of similar materials and hence there will be no coupling between the normal and tangential motion except that due to the influence of the coefficient of friction. Also, the surface displacements of the two solids due to a given traction distribution will be equal and opposite. The term displacement will hereafter always be taken to denote the combined (i.e. relative) displacement of the two surfaces.

2. The Normal Contact Problem

As there is no coupling between the normal and tangential motion, the classical Hertzian theory of impact can be used (see for example Timoshenko and Goodier [3]). In particular, the contact pressure at a radius r will be

$$\left.\begin{aligned} p(r) &= \frac{2G(b^2 - r^2)^{1/2}}{\pi R(1 - \nu)}; \quad 0 \le r \le b \\ &= 0; \qquad\qquad\qquad r > b, \end{aligned}\right\} \tag{1}$$

corresponding to a total force

$$P = \frac{4b^3 G}{3R(1 - \nu)}, \tag{2}$$

where b is the instantaneous radius of the contact area, R is the radius of the sphere and G, ν are respectively the modulus of rigidity and Poisson's ratio for the material.

The relative normal penetration of the solids is

$$u_z = b^2/R \tag{3}$$

and the normal component of relative velocity, obtained by solving the equation of motion, is

$$V_z = V_1(1 - cb^5)^{1/2}, \tag{4}$$

where

$$c = \frac{16G}{15MR^2V_1^2(1 - \nu)}, \tag{5}$$

V_1 is the value of V_z before the impact and M is the mass of the sphere.

3. Distribution of Tangential Traction

In their treatment of the oblique indentation problem, Mindlin and Deresiewicz [2] considered the loading history as the limit of a series of small increments of normal and tangential force applied successively. At each increment in tangential force, a small annulus of microslip is developed, but in the limit, when the series is replaced by an integral, no microslip region is predicted, provided the obliquity of loading does not exceed a certain critical value.

In effect, the increasing contact area permits new contact regions to be laid down in a deformed but traction free condition. A similar situation arises in the adhesive normal indentation of a half space by a rigid sphere (4) and (5). Once contact is established at any given region, no further deformation is possible during adhesive contact except for a rigid body displacement of the whole contact area.

It is therefore possible, following Mossakovskii [5], to express the tangential traction $f_x(r)$ at a radius r as an integral of that required to produce a tangential rigid body displacement of a circular region. A suitable integral representation is

$$\left.\begin{aligned} f_x(r) &= \frac{2G\mu}{\pi R(1-\nu)} \int_r^b \frac{sg(s)\,ds}{(s^2-r^2)^{1/2}}; & 0 \le r \le b \\ &= 0; & r > b, \end{aligned}\right\} \tag{6}$$

where $g(s)$ is an unknown function to be determined.

If $g(s)$ is independent of time (i.e. depends on radius only), the rate of change of traction at a given radius r will be

$$\frac{\partial f_x}{\partial t} = \frac{2G\mu b g(b)}{\pi R(1-\nu)(b^2-r^2)^{1/2}} \frac{db}{dt}, \tag{7}$$

and this differential traction can be shown to produce a differential relative displacement

$$\left.\begin{aligned} \frac{\partial u_x}{\partial t} &= \frac{(2-\nu)\mu b g(b)}{R(1-\nu)} \frac{db}{dt}; \\ \frac{\partial u_y}{\partial t} &= 0; \end{aligned}\right\} 0 \le r \le b \tag{8}$$

see, for example, Mindlin [6], Barber [7]. This expression is independent of radius, indicating that the traction distribution satisfies the condition for adhesive contact. (Note, however, that Eqn. (6) could be used to represent more general contact conditions if $g(s)$ were permitted to vary with time.)

An expression for db/dt can be found by differentiating Eqn. (3) with respect to time and substituting for V_z $(= du_z/dt)$ from Eqn. (4), giving

$$\frac{db}{dt} = \frac{RV_1}{2b}(1 - cb^5)^{1/2}. \tag{9}$$

Hence, during adhesive contact, the tangential relative velocity of the surfaces is

$$V_x = \frac{\partial u_x}{\partial t} = \frac{(2-\nu)\mu V_1 g(b)(1-cb^5)^{1/2}}{2(1-\nu)}, \tag{10}$$

from Eqns. (8) and (9).

4. The Equation of Tangential Motion

In order to set up the equation of tangential motion during impact, we need to integrate the traction distribution over the contact area to find the total tangential force, which is

$$F_x = \int_0^b 2\pi r f_x(r)\,dr = \frac{4G\mu}{R(1-\nu)} \int_0^b r\,dr \int_r^b \frac{sg(s)\,ds}{(s^2-r^2)^{1/2}}. \tag{11}$$

On reversing the order of integration and performing the inner integral, this reduces to

$$F_x = \frac{4G\mu}{R(1-\nu)} \int_0^b s^2 g(s)\, ds. \tag{12}$$

It follows that

$$\frac{dF_x}{db} = \frac{4G\mu b^2 g(b)}{R(1-\nu)}. \tag{13}$$

The tangential motion is determined by the equation

$$\frac{F_x}{M}\left(1 + \frac{R^2}{K^2}\right) = -\frac{dV_x}{dt} = -\frac{dV_x}{db}\cdot\frac{db}{dt}$$

$$= \frac{(2-\nu)RV_1^2\mu}{4(1-\nu)b}\,[\tfrac{5}{2}cb^4 g(b) - (1 - cb^5)^{1/2} g'(b)], \tag{14}$$

from Eqns. (9) and (10), where K is the radius of gyration of the sphere. Note that V_x is the relative tangential velocity local to the contact region and is compounded of a relative translation of the centres of mass and a relative rotation.

We can now derive a differential equation for the unknown function $g(b)$ by differentiating Eqn. (14) with respect to b and substituting for dF_x/db from Eqn. (13), obtaining

$$(1 - cb^5)b^2 g''(b) - (1 + 13cb^5/2)bg'(b) + 15cb^5(2\chi - 1)g(b)/2 = 0 \tag{15}$$

where

$$\chi = \frac{(1-\nu)}{(2-\nu)}\left(1 + \frac{R^2}{K^2}\right). \tag{16}$$

For a homogeneous solid sphere of a material with $\nu = 0.3$,

$$\chi = 1.44.$$

Finally, by making the substitutions

$$b = (x/c)^{1/5}, \tag{17}$$

$$g(b) = h(x), \tag{18}$$

Eqn. (15) is reduced to the standard form

$$10x(1-x)\frac{d^2h}{dx^2} + 3(2 - 7x)\frac{dh}{dx} + 3(2\chi - 1)h = 0. \tag{19}$$

This is the hypergeometric equation [8], whose two solutions are the functions

$$\left.\begin{aligned} h_1(x) &= F(\alpha, \beta; \gamma; x) \\ h_2(x) &= x^{2/5}F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; x) \end{aligned}\right\} \tag{20}$$

defined by the hypergeometric series

$$F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} \quad \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots \tag{21}$$

The parameters α, β, γ are here given by

$$\alpha, \beta = \frac{11 \pm \sqrt{240\chi + 1}}{20},$$

$$\gamma = \tfrac{3}{5}, \tag{22}$$

see Reference [8].

The series (21) converges in the range $|x| < 1$ and provides a practical algorithm for determining F for values of x as high as 0.95.

It follows from Eqns. (4) and (17) that x can be written in the form

$$x = 1 - (V_z/V_1)^2 \tag{23}$$

and hence $x = 1$ is a limiting value which is reached only at the instant of maximum penetration when V_z is zero. It will be shown below that complete adhesive contact terminates before this point is reached.

5. Initial Conditions

The general solution to Eqn. (19) can be constructed from an arbitrary linear combination of the two functions h_1, h_2 and hence two initial conditions are required to determine the corresponding constants. One is obtained from the condition of continuity of tangential velocity, V_x.

Writing

$$\psi = \frac{2(1-\nu)V_x}{\mu(2-\nu)V_1}, \tag{24}$$

we have

$$g(b) = \psi V_1/V_z, \tag{25}$$

from Eqns. (4) and (10) and hence, if the impact starts with a period of adhesive contact,

$$g(0) = \psi_1, \tag{26}$$

where ψ_1 is the value of ψ before the impact.

For small values of b, the tangential force (Eq. (12)) will be of the order b^3. It follows from Eqn. (14) that $g(b)$ must be of order b^4 or less and hence the coefficient of $h_2(x)$ in the solution must be zero if there is a period of adhesive contact at the start of an impact. The solution is therefore

$$g(b) = \psi_1 F(\alpha, \beta; \gamma; cb^5), \tag{27}$$

from Eqns. (17), (20), (21), and (26).

If the impact starts with a period of gross slip, the traction distribution during this period must be

$$f_x(r) = \frac{2G\mu(b^2 - r^2)^{1/2}}{\pi R(1 - \nu)}, \quad 0 \leq r \leq b \tag{28}$$

and the combined displacements of the solids due to this traction are easily shown to be

$$u_x = \frac{\mu}{8R(1 - \nu)}\{2(2 - \nu)(2b^2 - r^2) + \nu r^2 \operatorname{Cos} 2\theta\}, \tag{29}$$

$$u_y = \frac{\mu\nu r^2 \operatorname{Sin} 2\theta}{8R(1 - \nu)},$$

in the region $0 \leq r \leq b$, using the method of Reference [7].

Sliding will cease at a given point when $\partial u_x/\partial t$ at that point is equal to the local relative tangential velocity of the solids, V_x.

Differentiating Eqns. (29) with respect to time we have

$$\frac{\partial u_x}{\partial t} = \frac{\mu(2 - \nu)b}{R(1 - \nu)}\frac{db}{dt}; \qquad \frac{\partial u_y}{\partial t} = 0, \tag{30}$$

and hence sliding stops simultaneously at all points in the contact area when the tangential relative velocity has fallen to

$$V_x = \frac{\mu(2 - \nu)}{R(1 - \nu)} b \frac{db}{dt}, \tag{31}$$

i.e.

$$\psi = \frac{2b}{RV_1}\frac{db}{dt} = \frac{V_z}{V_1}, \tag{32}$$

from Eqns. (3) and (24).

During gross slip, the normal and tangential forces are proportional and it is readily shown from considerations of momentum that

$$\psi = \psi_1 - 2\chi(1 - V_z/V_1). \tag{33}$$

Hence, from Eqns. (32) and (33), the transition to adhesion must occur when

$$\frac{V_z}{V_1} = \left(\frac{\psi_1 - 2\chi}{1 - 2\chi}\right). \tag{34}$$

Note that the conditions for an initial period of adhesion ($\psi_1 < 1$) and for the whole impact to take place in gross slip ($\psi_1 \geq 4\chi - 1$, see Reference [1]) can be deduced from Eqn. (34) by putting $V_z = +V_1, -V_1$, respectively. An extended period of adhesion can only occur when the contact area is increasing and hence $V_z > 1$. This requires that the angle of incidence should satisfy the condition $\psi_1 < 2\chi$.

The distribution of traction (28) during gross slip can be expressed in the form of Eqn. (6) by writing $g(s) = 1$. It follows from Eqns. (10) and (14) that, for continuity in velocity and force, we must have

$$g(s) = 1; \qquad g'(s) = 0 \tag{35}$$

at the transition from gross slip to adhesion. The former condition could alternatively be derived from Eqns. (25) and (32). During the adhesive phase which follows, the traction distribution will be given by Eqn. (6) with $g(s) = 1$ for those values of s corresponding to contact regions laid down in the preceding gross slip phase.

Once the function $g(s)$ is known, the tangential velocity V_x and force F_x can be found from Eqns. (10) and (12) respectively. The results so obtained agree closely with those given by the numerical method [1].

6. End Conditions

As long as the value of $g(s)$ predicted by the above solution lies in the range $-1 < g(s) < +1$, the traction at any radius r predicted by Eqn. (6) will be numerically less than that required to produce slip, since the latter can be obtained from the same equation by writing $g(s) = +1$ or -1. Adhesive contact will therefore persist. However, the traction at the edge of the contact region ($r = b$) is entirely determined by the local value of $g(s)$ and hence as soon as a radius is reached for which $g(b) = 1$, microslip will start to occur there in the appropriate sense.

If $2\chi > 1$, the solutions defined by Eqns. (20) are both unbounded as $x \to 1$ and with the appropriate initial conditions we find that $g(b) \to -\infty$ as b approaches its maximum value. It follows that microslip always starts before the point of maximum penetration is reached and is in a sense opposed to the initial tangential motion.

For the special value $\chi = \frac{1}{2}$, the first solution of Eqn. (19) reduces to

$$h_1(x) = 1 \tag{36}$$

and the solution becomes

$$g(s) = \psi_1 \quad (\psi_1 \leq 1). \tag{37}$$

In each case, adhesive contact, once established, persists to the end of the impact. This situation is in fact a dynamic case of the constant direction oblique loading treated by Mindlin and Deresiewicz [2] and the condition for it to occur, $\chi = \frac{1}{2}$, effectively specifies a similar ratio of inertia to stiffness for tangential and normal motion.

If $\chi < \frac{1}{2}$, the tangential inertia is relatively greater than the normal and microslip, when it occurs, is in the same sense as the original motion. In this case, adhesion can only arise at the beginning of an impact, the condition of gross slip giving way immediately to microslip, even when the contact area is increasing.

Note, however, that a value of $\chi \leq \frac{1}{2}$ could only be achieved with an external flywheel, since even with $\nu = \frac{1}{2}$, this implies $K \geq \sqrt{2}R$ (see Eqn. (16)).

7. The Microslip Phase

The same method may be used to treat the subsequent phase of microslip. The central adhesive contact region (radius a) must still move as a rigid body, but in the slip region the traction is given by Eqn. (28). It follows that the traction distribution can be represented in the form of Eqn. (6) where $g(s)$ is given by the solution of the preceding adhesive phase for $s \leq a$ and $g(s) = -1$ for $a < s \leq b$. The tangential force and the velocity of the adhering region due to this traction distribution can then be found and substituted into the equation of motion as before. However, in this case, the only unknown is the inner radius of the slip region, a, and a differential equation is obtained in this quantity.

This differential equation proves to be non-linear and the author has so far been unable to produce an analytic solution. It would of course be possible to produce a numerical solution, but this would have little advantage over that described in Reference [1].

8. Conclusions

The above solution provides a rigorous analytic description of that phase of the impact of spheres during which there is complete adhesion between the surfaces. Such conditions are shown to arise at the beginning of the impact if $\psi_1 \leq 1$ or after an initial period of gross slip if $1 < \psi_1 < 2\chi$. In the latter case, the transition from gross slip to adhesion is instantaneous—i.e. the whole contact area comes relatively to rest at the same time.

The adhesive phase terminates before maximum penetration is reached, giving way to a state in which a surrounding annulus slips in the reverse sense to that holding at incidence for most practical systems.

The analytical solution obtained agrees closely with a previous numerical treatment of the problem.

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Abstract

An integral representation is developed for the distribution of tangential traction during adhesive contact between obliquely loaded elastic spheres of similar materials. It is shown that adhesive conditions arise at the beginning of an impact between two spheres at low angles of incidence or after an initial period of gross slip for larger angles.

A differential equation is developed and solved, describing this phase of the impact.

Résumé

On établit une représentation intégrale de la distribution de traction tangentielle pendant le contact adhésif entre deux sphères de matériaux élastiques similaires chargées obliquement. On démontre que des conditions adhésives s'établisseut au commençement d'un choc entre deux sphères aux petits angles d'incidence ou après une période initiale de glissement aux angles plus grands.

Une équation différentielle décrivant cette phase du choc est élaborée et résolue.

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