Journal of Applied Mathematics and Physics (ZAMP)
Vol. 27, 1976 Birkhäuser Verlag Basel

The Effect of Heat Flow on the Contact Area between a Continuous Rigid Punch and a Frictionless Elastic Half-space

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1. Introduction

Consider the problem of a rigid frictionless punch pressed into the plane surface of an elastic half-space. If the punch surface is continuous, the extent of the contact region is not known *a priori*, but has to be determined from two inequalities stating that contact pressure is always positive and that the regions occupied by the two solids do not overlap. Some general features of the isothermal problem have been discussed in a previous paper [1].

Suppose that the temperature of the punch is now raised so that heat is conducted into the half-space through the contact area, inducing a state of thermal stress. We should expect regions near the contact area to expand, causing the two solids to separate if the compressive load remains constant. The separation might also be expected to cause a reduction in the extent of the contact area between the punch and the half-space.

This result has been confirmed experimentally [2] and theoretically [3, 4] for the particular case of two solids with spherical surfaces in contact over a circular region.

In this paper, it is shown to be true for a rigid punch of any continuous shape indenting an elastic half-space. It is also shown that the contact area cannot be multiply-connected if the punch surface is convex. This result can be used (for example) to justify the assumption of a single circular contact area in axisymmetric thermoelastic contact problems [5].

2. Preliminary Results

We shall require the following well known theorems of potential theory:

Lemma 1. *If a function V is harmonic in a space bounded by a surface S, the maximum value of V must occur on S at a point where $\partial V/\partial n < 0$, n being the inward drawn normal to S.*

The maximum could only occur within the enclosed space at a point where $\partial^2 V/\partial x^2$, $\partial^2 V/\partial y^2$, $\partial^2 V/\partial z^2$ were separately negative [6], but no such point exists since

$\nabla^2 V = 0$. The maximum must therefore occur on the boundary at some point P and the value of V within the space must be everywhere less than the maximum. Hence the potential gradient away from P into the space must be negative.

The same argument can be used to prove

Lemma 2. *If the maximum value of V occurs over an extended region S_1 of S, the potential gradient $\partial V/\partial n < 0$ throughout S_1.*

Note: When the surface S extends to infinity, it is possible for the maximum or the minimum potential (but not both) to occur there.

3. Representation of the Stress State

We shall apply these results to problems of frictionless steady-state thermoeleastic contact, making use of a solution of the thermoelastic equations due to Williams [7]. The displacement vector $\underline{u}$ is expressed in the form

$$\underline{u} = z\nabla \frac{\partial \phi}{\partial z} - \nabla\phi + \underline{k}\frac{\partial \phi}{\partial z} - (1 - 2\nu)\nabla\psi - z\nabla\frac{\partial \psi}{\partial z} + (3 - 4\nu)\underline{k}\frac{\partial \psi}{\partial z}, \tag{1}$$

where the temperature rise above a suitable datum is

$$T = \frac{2(1 - \nu)}{\alpha(1 + \nu)}\frac{\partial^2 \phi}{\partial z^2}, \tag{2}$$

ν, α are respectively Poisson's ratio and the coefficient of thermal expansion for the material and ϕ, ψ are two harmonic potential functions. The unit vector in the z direction is denoted by $\underline{k}$.

The half-space is considered to occupy the region $z \geq 0$.

The normal component of displacement at its plane surface $z = 0$ reduces to

$$u_z = 2(1 - \nu)\frac{\partial \psi}{\partial z}, \tag{3}$$

whilst the stress acting on that surface is a purely normal traction

$$\sigma_{zz} = 2G\frac{\partial^2}{\partial z^2}(\psi - \phi), \tag{4}$$

where G is the modulus of rigidity of the material.

If we write p $(= -\sigma_{zz})$ for normal contact pressure, equations (2, 4) give

$$p - G\frac{\alpha(1 + \nu)T}{(1 - \nu)} = -2G\frac{\partial^2 \psi}{\partial z^2}. \tag{5}$$

The heat flow into the surface is given by

$$q_z = -K\frac{\partial T}{\partial z} = -\frac{2K(1 - \nu)}{\alpha(1 + \nu)}\frac{\partial^3 \phi}{\partial z^3}, \tag{6}$$

from equation (2), where K is the thermal conductivity of the material of the half-space.

4. Application to Contact Problems

Theorem 1. *If the punch surface is everywhere convex, the contact area cannot be multiply-connected, whether the punch be heated or cooled.*

Proof. We represent the state of stress in the half-space in terms of the two potential functions ϕ, ψ as defined in section 3 above. The contact area is denoted by A and the rest of the surface of the half-space by $\bar{A}$.

Tensile contact stresses are inadmissible and hence, at the surface $z = 0$, we have

$$\frac{\partial^2\psi}{\partial z^2} - \frac{\partial^2\phi}{\partial z^2} \leq 0 \tag{7}$$

from equation (4).

The function $[(\partial^2\psi/\partial z^2) - (\partial^2\phi/\partial z^2)]$, being made up of the derivatives of harmonic functions, is itself harmonic and by Lemma 1, its maximum value must occur on the surface and is therefore zero. Furthermore, throughout $\bar{A}$ we have

$$\frac{\partial^2\psi}{\partial z^2} - \frac{\partial^2\phi}{\partial z^2} = 0 \tag{8}$$

and hence by Lemma 2,

$$\frac{\partial^3\psi}{\partial z^3} - \frac{\partial^3\phi}{\partial z^3} < 0. \tag{9}$$

We shall assume that there is no heat loss from the exposed surfaces in which case $\partial^3\phi/\partial z^3 = 0$ (from equation (6)) and

$$\frac{\partial^3\psi}{\partial z^3} < 0 \tag{10}$$

throughout $\bar{A}$.

Now suppose the theorem is false, in which case there exists a totally enclosed region of $\bar{A}$ in which

$$u_z - u > 0 \tag{11}$$

where u is a prescribed function of position in the plane $z = 0$ representing the profile of the punch.

The function $(u_z - u)$ tends to zero at the boundary of this region and is positive within it. There must therefore be some internal point at which $(u_z - u)$ is a local maximum and hence

$$\nabla_1^2(u_z - u) \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u_z - u) < 0. \tag{12}$$

Since the punch is convex ($\nabla^2 u < 0$) it follows that there exists a point in $\bar{A}$ at which $\nabla_1^2 u_z < 0$ and hence, from equation (3),

$$\frac{\partial^3 \psi}{\partial z^3} > 0 \tag{13}$$

contradicting condition (10).

We therefore conclude that there can be no totally enclosed region in $\bar{A}$, i.e., $\bar{A}$ is connected and A is not multiply-connected.

Punch Loaded and then Heated

Suppose that a rigid punch is pressed into the half-space under isothermal conditions, establishing a contact area A_1. The remainder of the surface of the half-space is denoted by $\bar{A}_1$.

The punch is now heated, causing thermal distortion of the half-space and establishing a different contact area A_2.

If the heat flow in the punch is similar to that in the half-space—i.e., if the contact area is small in comparison with the other dimensions of the punch—the temperature will be uniform throughout A_2. It will be denoted by T_0 relative to the temperature at the extremities of the half-space. We assume that there is no heat loss from the non-contact region $\bar{A}_2$ and hence the temperature throughout this region must lie in the range $0 \leq T < T_0$.[1])

The surface of the half-space is made up of the four regions $A_1 \cap A_2$; $\bar{A}_1 \cap A_2$; $A_1 \cap \bar{A}_2$; $\bar{A}_1 \cap \bar{A}_2$. Certain general statements can be made about the changes in contact pressure and displacement in these regions when the punch is heated. The contact pressure p must increase in those regions which make contact, fall in those which leave contact and remain unchanged (at zero) outside both contact regions. The change in displacement, δu_z, will be constant throughout $A_1 \cap A_2$ and equal to the rigid body displacement of the punch relative to the extremities of the half-space (denoted by u_0). Geometrical considerations demand that, in any new contact region, $\delta u_z < u_0$, whilst in any region in which contact is lost, $\delta u_z > u_0$. These results are summarized in Table 1.

Theorem 2. *Heating the punch causes its indentation to decrease* (i.e. $u_0 < 0$).

Proof. We consider the state of stress corresponding to the difference between the heated punch problem and the isothermal problem. This *differential* problem can be described in terms of two harmonic potential functions ϕ, ψ as defined in section 3 above.

The function $\partial\psi/\partial z$ is harmonic and hence, by Lemma 1, its maximum value must occur on the surface in a region where $\partial^2\psi/\partial z^2 < 0$ or at infinity. Since the maximum occurs on the surface, it must coincide with the maximum value of δu_z (see equation 3).

[1]) This condition would also be satisfied if there was radiation at $\bar{A}_2$ into a medium at a temperature within the stated range.

Table 1

			Change in contact pressure p	Change in normal displacement u_z	Change in temperature T
(a)	$A_1 \cap A_2$	remaining in contact		$\delta u_z = u_0$	$\delta T = T_0$
(b)	$\bar{A}_1 \cap A_2$	new contact region	$\delta p > 0$	$\delta u_z < u_0$	$\delta T = T_0$
(c)	$A_1 \cap \bar{A}_2$	region leaving contact	$\delta p < 0$	$\delta u_z > u_0$	$T_0 > \delta T > 0$
(d)	$\bar{A}_1 \cap \bar{A}_2$	remaining out of contact	$\delta p = 0$		$T_0 > \delta T \geq 0$

The non-contact region (d) includes the region at infinity in which u_z, δp, $\delta T \to 0$.

From the constraints on δu_z (Table 1), we conclude that the maximum can only occur in (b) if $A_1 = 0$, which is impossible, and can only occur in (a) if $A_1 \cap \bar{A}_2 = 0$, i.e., if there are no regions leaving contact.

Furthermore, it cannot occur in regions (c, d), (excluding the region at infinity), since there we have

$$\delta\left\{p - \frac{G\alpha(1+\nu)T}{(1-\nu)}\right\} < 0 \tag{14}$$

(see Table 1), and hence $\partial^2\psi/\partial z^2 > 0$ from equation (5).

We conclude that *either* the maximum δu_z occurs in (a) and $A_1 \cap \bar{A}_2 = 0$, *or* it occurs at infinity.

If the first alternative were realised, the maximum value would exist throughout (a) and hence by Lemma 2 and equation (5) we should have

$$\delta\left\{p - \frac{G\alpha(1+\nu)T}{(1-\nu)}\right\} > 0 \tag{15}$$

throughout (a). But $T = T_0$ in (a) and hence this condition implies that $\delta p > 0$ throughout (a), which is impossible in the absence of any reduction in the contact area, since the total compressive force is unchanged.

We must therefore conclude that the maximum value of δu_z occurs at infinity and is zero. Hence, in the finite domain

$$\delta u_z < 0 \tag{16}$$

and in particular, $u_0 < 0$, i.e., the solids move further apart.

An exactly parallel argument can be used to show that the solids move together ($u_0 > 0$) if the punch is cooled.

Theorem 3. *If the punch surface is continuous, some regions will lose contact when it is heated* (i.e., $A_1 \cap \bar{A}_2 \neq 0$).

Proof. By Lemma 1, the maximum value of the harmonic function $\partial^2\psi/\partial z^2$ must occur in a region of the surface at which $\partial^3\psi/\partial z^3 < 0$, where ψ relates to the differential problem as defined in the proof of theorem 2. Since $\partial\psi/\partial z$ is harmonic it follows that

$$\nabla_1^2 \frac{\partial\psi}{\partial z} = \frac{1}{2(1-\nu)} \nabla_1^2 \delta u_z > 0 \tag{17}$$

at this point.

Now suppose that there are no regions leaving contact ($A_1 \cap \bar{A}_2 = 0$).

The total load is unchanged and hence there must be some parts of A_1 where $\delta p \leq 0$ and hence where

$$-\delta\left\{p - \frac{G\alpha(1+\nu)T}{(1-\nu)}\right\} = 2G\frac{\partial^2\psi}{\partial z^2} \geq \frac{G\alpha(1+\nu)T_0}{(1-\nu)}. \tag{18}$$

Throughout $\bar{A}_1$, on the other hand, we have $\delta p \geq 0$, $\delta T < T_0$ and hence

$$-\delta\left\{p - \frac{G\alpha(1+\nu)T}{(1-\nu)}\right\} < \frac{G\alpha(1+\nu)T_0}{(1-\nu)}. \tag{19}$$

It follows that the maximum value of $\partial^2\psi/\partial z^2$ must occur in A_1 and hence in $A_2 \cap A_1$, since $\bar{A}_2 \cap A_1 = 0$ *ex hypothesi.*

However, δu_z is constant through $A_2 \cap A_1$ and

$$\nabla_1^2 u_z = 0, \tag{20}$$

contradicting condition (17). We therefore conclude that the hypothesis $A_1 \cap \bar{A}_2 = 0$ is untenable and hence that some regions lose contact when the punch is heated.

A similar argument can be developed to show that some new contact areas must be established if the punch is cooled.

We note that condition (20), may be suspended at the boundary of the region $A_1 \cap A_2$ if the displacement there is discontinuous in the first derivative. This is only possible if there is a corresponding discontinuity in the surface of the punch.

5. Conclusions

The above theorems have shown that if a continuous rigid punch indenting a frictionless elastic half-space is heated, the separation of the solids will increase and part of the contact area will be lost. It seems probable that no new contact regions would be established, but the author has been unable to find a proof of this result. It would also be of interest to know whether the separation and the extent of the contact area necessarily change monotonically with the temperature of the punch. The readers' attention is drawn to these as yet unsolved problems.

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Abstract

A general proof is given of the theorem that, if a continuous rigid punch indenting an elastic half space is heated, the separation of the solids will increase and part of the contact area will be lost. It is also shown that if the punch is convex, the contact area cannot be multiply-connected.

Résumé

On donne une preuve générale du théorème que, si un poinçon continu rigid, pénétrant un demi-espace élastique, est chauffé, la séparation des solides grandira et une partie de la région de contact sera perdue. On montre également que la région de contact ne peut pas être multiplement connexe si le poinçon est convexe.

(Received: March 30, 1976)