Exact Expressions for the Roots of the Secular Equation for Rayleigh Waves

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1 Introduction

The speed c_R at which Rayleigh waves can propagate over the surface of an isotropic linear elastic half-space is a root of the equation

$$R(V) \equiv \left(2 - M_2^2\right)^2 - 4\sqrt{\left(1 - M_1^2\right)\left(1 - M_2^2\right)} = 0, \quad (1)$$

where

$$M_1 = \frac{V}{c_1}; \quad M_2 = \frac{V}{c_2}, \quad (2)$$

$c_1 = \sqrt{(\lambda + 2\mu)/\rho}$, $c_2 = \sqrt{\mu/\rho}$ are the dilatational and shear wave speeds, respectively, λ, μ are Lamé's constants and ρ is the density.

Multiplying Eq. (1) by the expression $(2 - M_2^2)^2 + 4\sqrt{(1 - M_1^2)(1 - M_2^2)}$, of which the only real zero is $V = 0$, and cancelling the factor M_2^2 corresponding to this trivial root, we obtain the equation

$$m^3 - 8m^2 + (24 - 16\Lambda)m - 16(1 - \Lambda) = 0 \quad (3)$$

where $m = M_2^2$,

$$\Lambda = \frac{c_2^2}{c_1^2} = \frac{(1 - 2\nu)}{2(1 - \nu)} \quad (4)$$

and ν is Poisson's ratio. We note that for the range $-1 \le \nu \le 0.5$, $0 \le \Lambda \le 0.75$.

Equation (3) is a cubic equation in m and can therefore be solved explicitly by standard methods. In view of the simplicity of this procedure, it is remarkable that the resulting closed-form solutions are not given in any of the standard reference works on Elastodynamics, (e.g., Achenbach, 1984; Aki and Richards, 1980; Auld, 1973; Ben Menahem and Singh, 1981; Brekhovskikh and Godin, 1990; Bullen, 1962; Cagniard, 1964; Eringen and Suhubi, 1975; Ewing et al., 1957; Fedorov, 1968; Hanyga, 1985; Jeffreys, 1952; Kolsky, 1952; Love, 1944; Mal and Singh, 1991; Viktorov, 1967), except for the special cases $\nu = 0$, 0.25, nor as far as the present authors have been able to ascertain are they available elsewhere in the literature. The purpose of the present Note is therefore to develop the closed-form solutions of (3), and hence of the Rayleigh Eq. (1) for general values of ν.

Of course, numerical solutions for c_R are easily obtained to any desired accuracy and are widely available in the literature. However, apart from the aesthetic appeal of a closed-form solution, we note that the Rayleigh function $R(V)$ appears widely in the solution of classical elastodynamic problems. For example, it appears in the denominator of Lamb's solution (1904) for an impulsive normal force on the surface of a half-space and in Cole and Huth's solution (1958) for a line load moving steadily over the surface of the half-plane. Both these results can be used as Green's functions to generate convolution integrals for more general boundary value problems and the availability of an explicit factorization of the rationalized form (3) of $R(V)$ permits such integrals to be broken down into simpler terms by partial fractions.

2 Solution

Equation (3) can be reduced to the standard form

$$x^3 + px + q = 0 \quad (5)$$

by the substitution

$$x = m - \frac{8}{3} \quad (6)$$

where

$$p = \frac{8}{3}(1 - 6\Lambda); \quad q = \frac{16}{27}(17 - 45\Lambda). \quad (7)$$

2.1 Nature of the Roots.

The sign of the discriminant

$$D = -\left(\frac{q^2}{4} + \frac{p^3}{27}\right) \quad (8)$$

of Eq. (5) determines the nature of the three roots. In particular,

(1) if $D > 0$, Eq. (5), and hence Eq. (3), has three distinct real roots.
(2) if $D = 0$, the equation has three real roots, at least two of which are equal.
(3) if $D < 0$, the equation has one real and two complex conjugate roots.

Substituting (7) into (8), it is easily verified that $D(\Lambda)$ is negative at $\Lambda = 0$ and changes sign once in the range $0 \le \Lambda \le 0.75$.

The exact value of Λ at which this sign change occurs is the root of the equation $D(\Lambda) = 0$, which can be written in the form

$$\Lambda^3 - \frac{107}{64}\Lambda^2 + \frac{31}{32}\Lambda - \frac{11}{64} = 0. \quad (9)$$

This is also a cubic equation and it can be converted into the form (5) by the substitution

$$\Lambda = x + \frac{107}{192} \quad (10)$$

in which case

$$p^* = \frac{455}{12288}; \quad q^* = \frac{77293}{3538944} \quad (11)$$

where the *s are used to distinguish these quantities from those defined in Eq. (7). Substituting these results into (8), we find that $D^* < 0$ and hence deduce that Eq. (9) has only one real root. The value of this root is obtained from Cardan's formula (Cowles and Thompson, 1947) as

$$x^* = \sqrt[3]{-\frac{q^*}{2} + \sqrt{-D^*}} + \sqrt[3]{-\frac{q^*}{2} - \sqrt{-D^*}}. \quad (12)$$

Substituting for p^*, q^*, D^* from Eqs. (11), (8) and using (10), we find that the corresponding critical value of Λ is

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$$\Lambda^* = \frac{107}{192} + \sqrt[3]{-\frac{77293}{7077888} + \sqrt{\frac{6859}{56623104}}} + \sqrt[3]{-\frac{77293}{7077888} - \sqrt{\frac{6859}{56623104}}} = 0.3214984\ldots \quad (13)$$

Solving Eq. (4) for ν, we find

$$\nu = \frac{(1-2\Lambda)}{2(1-\Lambda)} \quad (14)$$

and the value corresponding to Λ^* is

$$\nu^* = 0.2630821\ldots \quad (15)$$

Since Λ^* is the only real root of (9) we conclude that

(1) for $-1 < \nu < \nu^*$, Eq. (3) has three distinct real roots.
(2) for $\nu = \nu^*$, Eq. (3) has three real roots, two of which are equal.
(3) for $\nu^* < \nu^* < 0.5$, Eq. (3) has one real root (one of which is the Rayleigh root) and a pair of complex conjugate roots.

Hayes and Rivlin (1962) report that a similar conclusion was reached by Somigliana, who gave the critical value as $\nu^* = 0.2637$.

2.2 Expressions for the Roots. We now proceed to determine explicit expressions for the roots in these three ranges.

Case 1: $-1 < \nu < \nu^*$.

For this case, the three roots of Eq. (3) are all real and can be written in the form

$$m_1 = \frac{8}{3} + 2\sqrt{-\frac{p}{3}}\cos\left(\phi + \frac{2\pi}{3}\right) \quad (16)$$

$$m_2 = \frac{8}{3} + 2\sqrt{-\frac{p}{3}}\cos\left(\phi + \frac{4\pi}{3}\right) \quad (17)$$

$$m_3 = \frac{8}{3} + 2\sqrt{-\frac{p}{3}}\cos\phi \quad (18)$$

(see Cowles and Thompson, 1947), where

$$\phi = \frac{1}{3}\arccos\left(\frac{3q}{2p\sqrt{-p/3}}\right). \quad (19)$$

If we define the principal value of $\arccos(x)$ in Eq. (19) to lie in the range $0 \le x \le \pi$, it follows that $m_1 < m_2 < m_3$. The root corresponding to the Rayleigh wave speed is m_1—i.e.,

$$c_R = c_2\sqrt{m_1} \quad (20)$$

and the other two roots both correspond to speeds higher than the dilatational wave speed—i.e., m_2, $m_3 > M_1^2$ for all ν. Hayes and Rivlin (1962) show that the elastodynamic solutions corresponding to these roots involve physically unacceptable unbounded fields at infinity.

Case 2: $\nu = \nu^*$.

This can be regarded as a limit of the preceeding case, in which $\phi \to \frac{\pi}{3}$ and m_2, m_3 become equal. The results are

$$m_1 = \frac{8}{3} + 2\sigma \quad (21)$$

$$m_2 = m_3 = \frac{8}{3} - \sigma \quad (22)$$

where

$$\sigma = \frac{2}{3}\sqrt[3]{\frac{11-56\nu^*}{2(1-\nu^*)}} \quad (23)$$

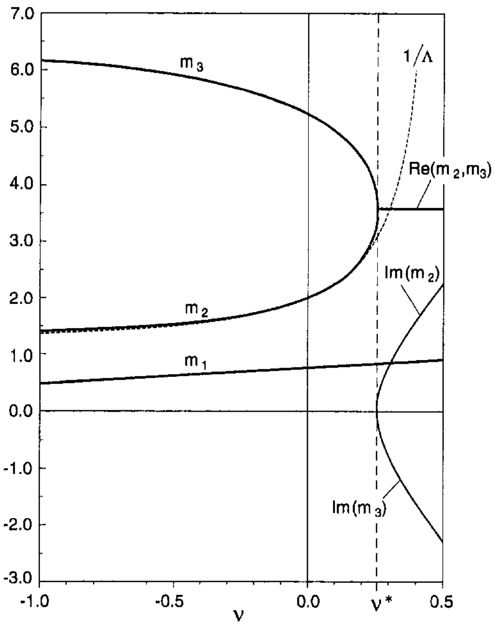

Fig. 1 The three roots, m_1, m_2, and m_3, as functions of ν

Case 3: $\nu^* < \nu < 0.5$.

In this case only the Rayleigh root m_1 is real and the three roots of Eq. (3) are given by Cardan's formula as

$$m_1 = \frac{8}{3} + \gamma + \eta \quad (24)$$

$$m_2 = \frac{8}{3} - \frac{\gamma+\eta}{2} + i\frac{\sqrt{3}}{2}(\gamma - \eta) \quad (25)$$

$$m_3 = \frac{8}{3} - \frac{\gamma+\eta}{2} - i\frac{\sqrt{3}}{2}(\gamma - \eta) \quad (26)$$

where

$$\gamma = \frac{2}{3}\sqrt[3]{-(17-45\Lambda) + \sqrt{(17-45\Lambda)^2 + 8(1-6\Lambda)^3}} \quad (27)$$

$$\eta = \frac{2}{3}\sqrt[3]{-(17-45\Lambda) - \sqrt{(17-45\Lambda)^2 + 8(1-6\Lambda)^3}} \quad (28)$$

and Λ is defined by Eq. (4).

3 Conclusion

This completes the solution for the roots of Eq. (3). In each case a closed-form expression for the Rayleigh wave speed can be written in the form

$$c_R = c_2\sqrt{m_1}. \quad (29)$$

The three roots, m_1, m_2, m_3 are shown as functions of ν in Fig. 1. It is notable that the second root, m_2, is very close to the function $1/\Lambda$, when ν is small. In fact, it can be shown that

$$m_2 = \frac{1}{\Lambda} + 32\epsilon^4 + O(\epsilon^5) \tag{30}$$

for $\nu \ll 1$, where

$$\epsilon = \frac{1}{2} - \Lambda = \frac{\nu}{2(1-\nu)}. \tag{31}$$

We also note that the results permit the left-hand side of (3) to be factorized explicitly in the form $(m - m_1)(m - m_2)(m - m_3)$. It follows that the function $R(V)^{-1}$, which appears in Lamb's solution (1904) for an impulsive load and Cole and Huth's solution (1985) for a moving line load, can be written in the rationalized form

$$\frac{1}{R(V)} = \frac{\left(2 - M_2^2\right)^2 + 4\sqrt{\left(1 - M_1^2\right)\left(1 - M_2^2\right)}}{M_2^2\left(M_2^2 - m_1\right)\left(M_2^2 - m_2\right)\left(M_2^2 - m_3\right)} \tag{32}$$

which can be expanded as a set of partial fractions if desired.

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Critical Strain Ranking of 12 Materials in Deformations Involving Adiabatic Shear Bands

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Batra and Kim (1992) studied the initiation and growth of shear bands in 12 materials deformed in simple shear. Each material was modeled by the Johnson-Cook (1983) law, values of material parameters were taken from Johnson and Cook's (1983) paper, and the effects of inertia forces and thermal conductivity were included. However, materials generally are rarely tested in simple shear and the material data used was derived from tests conducted over a moderate range of strains, strain rates, and temperatures. In this Note, we report results of numerical simulation of torsion tests similar to those performed by Marchand and Duffy (1988) on HY-100 steel and rank 12 materials according to the values of the nominal strain at which the torque begins to drop precipitously. Values of material parameters taken from Rajendran's report (1992) and likely to be valid over a large range of strains, strain rates, and temperatures are used. However, the effect of thermal conductivity has been neglected because the computer code DYNA3D (Whirley and Hallquist, 1991) employed to study the problem assumes locally adiabatic deformations. Batra and Kim (1991) have shown that for simple shearing deformations of viscoplastic materials, realistic values of thermal conductivity have little effect on the values of the nominal strain at which deformations begin to localize and thus shear bands initiate.

In the simulations reported herein, the initial thickness, $\omega(z)$, of the tube with inner radius of 4.75 mm is assumed to vary according to the relation

$$\omega(z) = 0.19\left[1.9 + 0.1\sin\left(\frac{1}{2} + \frac{2z}{2.5}\right)\pi\right]\text{mm}, \qquad 0 \le z \equiv 2.5\text{ mm}. \tag{1}$$

Here z denotes the position of a point along the axis of the tube with $z = 0$ being the fixed end. The end $z = 2.5$ mm is twisted so as to produce a nominal strain rate of $5000\ \text{s}^{-1}$. It is assumed that the angular speed increases from zero to the steady value of 2530 rad/s in 20 μs. The thickness variation, depicted in Fig. 1, clearly shows that the minimum tube thickness occurs at its center, and equals 90 percent of that at its outer edges.

The tube is assumed to be initially at rest, stress-free, and at the room temperature, T_o, of 25°C. The inner and outer surfaces of the tube are taken to be traction-free and thermally insulated, and its deformations are assumed to be

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