International Journal of Solids and Structures 48 (2011) 2041–2047

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Frictional elastic contact with periodic loading

J.R. Barber [a,*], M. Davies [b,c], D.A. Hills [c]

[a] Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109-2125, USA
[b] Rolls-Royce plc, P.O. Box 31, Derby, DE24 8BJ, UK
[c] Department of Engineering Science, Oxford University, Parks Road, Oxford, UK

ARTICLE INFO

Article history:
Received 14 December 2010
Received in revised form 3 March 2011
Available online 16 March 2011

Keywords:
Contact problems
Fretting fatigue
Shakedown
Coulomb friction
Hysteretic damping

ABSTRACT

Quasi-static frictional contact problems for bodies of fairly general profile that can be represented as half planes can be solved using an extension of the methods of Ciavarella and Jäger. Here we consider the tangential traction distributions developed when such systems are subjected to loading that varies periodically in time. It is shown that the system reaches a steady state after the first loading cycle. In this state, part of the contact area (the *permanent stick zone*) experiences no further slip, whereas other points may experience periods of stick, slip and/or separation. We demonstrate that the extent of the permanent stick zone depends only on the periodic loading cycle and is independent of the initial conditions or of any initial transient loading phase. The exact traction distribution in this zone *does* depend on these factors, but the resultant of these tractions at any instant in the cycle does not. The tractions and slip velocities at all points outside the permanent stick zone are also independent of initial conditions, confirming an earlier conjecture that the frictional energy dissipation per cycle in such systems depends only on the periodic loading cycle. We also show that these parameters remain unchanged if the loading cycle is changed by a time-independent tangential force, provided this is not so large as to precipitate a period of gross slip (sliding).

1. Introduction

If an engineering system comprises elastic components with frictional interfaces, the instantaneous state generally depends on the previous loading history. In particular, if the system is subjected to loads that are periodic in time, the steady state may depend on the initial conditions or on an initial transient loading phase. As a result, the effective damping in the system and/or damage due to fretting may exhibit variability due to differences in assembly procedures (Barber, 2011).

It has recently been established that a frictional version of Melan's theorem (Melan, 1936) applies to such systems if and only if the tangential and normal contact problems are uncoupled (Barber et al., 2008). In other words, if a change in slip displacements has no effect on the normal contact tractions, and if there exists a set of initial conditions such that the entire contact region can remain in a state of 'stick' without the tractions violating the limiting Coulomb friction conditions at any point, then the system will shake down to such a state regardless of the initial conditions. For coupled systems, counter examples can always be found in which there exists such a 'safe shakedown state' for a given set of periodic loads, which however is not reached from certain combinations of initial conditions (see for example, Ahn et al., 2008).

One conclusion of the frictional Melan's theorem is that if there exists a safe shakedown state, the steady-state response of an uncoupled frictional system depends on the initial conditions only to the extent of a set of locked-in slip displacements in the contact zone. It is conjectured (Barber, 2011) that this is a special case of a more general but unproven result that even above the shakedown limit the steady-state response depends on the initial conditions only to the extent of a similar set of slip displacements that are now restricted to a unique 'permanent stick zone' – i.e. a part of the contact area that does not slip during the steady state. This could be seen as the frictional counterpart of the theorem in associative plasticity that the time-varying terms in the stresses and strains and the extent of the zone experiencing cyclic plasticity are independent of the pre-existence of any self-equilibrated state of residual stress (Polizzotto, 2003).

In this paper, we shall establish this conjecture for all two-dimensional frictional elastic contact problems for which the bodies may be adequately described as half planes subjected to periodic remote loading and for which the normal and tangential contact problems are uncoupled, meaning that the normal contact tractions are uninfluenced by the relative slip displacements. We shall also develop techniques for determining the extent of microslip zones and tangential tractions in fairly general periodic loading

* Corresponding author. Tel.: +1 7349360406; fax: +1 7346156647.
E-mail address: jbarber@umich.edu (J.R. Barber).

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doi:10.1016/j.ijsolstr.2011.03.008

problems of this class and establish an additional general result that these quantities are independent of the mean tangential load as long as this does not cause gross slip (sliding) at any time during the cycle.

We should first remark that though Klarbring (1999) has established a condition under which the discrete quasi-static frictional contact problem is well posed, no such result is available for the corresponding general continuum problem. However, for the restricted class of problems comprising the frictional contact of two elastic half planes, the contact tractions and displacements are related by Cauchy singular integral equations (Hills et al., 1993) and the evolutionary solution can be shown to exist and be unique for all values of the coefficient of friction, using the properties of the solutions of these integrals. Also, the solution procedure outlined in the present paper will be shown to generate a traction and displacement field satisfying the Coulomb friction law for half planes of arbitrary profile and any loading scenario, so existence is effectively proved for this class of problems by exhaustion.

The Cauchy integral formulation can also be used to show that the normal and tangential contact problems are uncoupled in the sense defined above as long as Dundurs' bimaterial constant $\beta = 0$, since, for example, the integral equation satisfied by the normal contact tractions then depends only on the applied normal load and the initial profile of the contacting bodies and is unaffected by tangential contact tractions (Barber, 2010, Section 12.7). This condition is satisfied by virtue of symmetry if the materials of the contacting bodies are similar.

1.1. The Cattaneo–Mindlin problem

Investigation of problems of this class dates back to a seminal paper by Cattaneo (1938) [see also Mindlin (1949)], who gave a solution for the problem in which a classical Hertzian contact is first loaded by a normal force P and then by an increasing tangential force Q. The resulting tangential tractions can be written as the superposition of the 'full slip' tractions over the entire contact area and a similar distribution over a smaller central stick area. Ciavarella (1998a) and Jäger (1997, 1998) showed that this form of superposition is not restricted to Hertzian contact, but applies to all two-dimensional contact problems provided the bodies can be approximated as half planes and $\beta = 0$. A brief explanation of the basis of their argument is given in Section 2 below.

A generic problem of this class is illustrated in Fig. 1. However, notice that the Ciavarella–Jäger theorem imposes no restrictions on the profiles of the contacting bodies and that the resulting contact area need not necessarily be connected. For example, the theorem applies equally to the contact of rough surfaces of specified profile, where contact will occur in a set of microscopic 'actual contact areas' (Dini and Hills, 2009). It also applies in an approximate sense when the contact is three-dimensional (the same approximation applies to Cattaneo and Mindlin's original three-dimensional solutions.

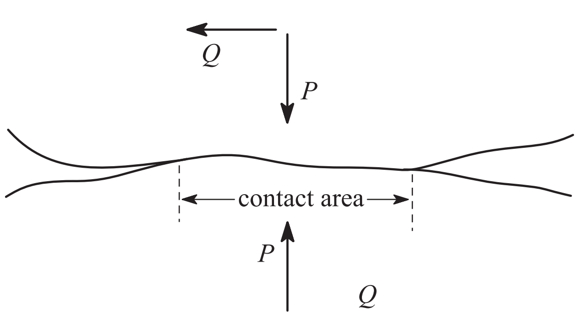

Fig. 1. A generalized plane contact problem.

The Cattaneo traction distribution is restricted to the case where the normal and tangential loads are applied in sequence, but similar techniques can clearly be applied to at least some other cases. Mindlin and Deresiewicz (1953) considered a range of cases where a further finite increment of loading is applied and showed that full stick would occur if the increments ΔP, ΔQ in normal and tangential force respectively satisfied the condition $|\Delta Q| < f\Delta P$, where f is the coefficient of friction. In other cases, the traction distribution can be obtained by a further superposition of a Cattaneo–Mindlin distribution for the loading increment. This technique was extended to general geometries by Jäger (1998), who also developed an algorithm for considering more general loading scenarios as a sequence of straight line segments in PQ-space.

1.2. Periodic loading

We consider the case where the contact is subject to a combination of mean and periodic loading, typically as a result of externally driven vibration. For example, the loading might take the form

$$P(t) = P_0 + P_1 \cos(\omega t); Q(t) = Q_0 + Q_1 \cos(\omega t + \phi), \quad (1)$$

where t is time and ϕ is a relative phase angle.

This scenario will be represented as an ellipse in PQ-space, as shown in Fig. 2. The entire periodic loading cycle must lie between the limiting lines $Q = \pm fP$ if there is to be no gross slip (sliding). We also assume that the system starts from the unloaded condition $P = Q = 0$ and hence some initial loading phase OA is needed to reach the periodic state. We shall show later that this initial phase affects the tangential tractions in the permanent stick zone – i.e. in that part of the contact area that does not slip during the steady state – but it has no other effect on the steady-state problem and in particular does not affect the frictional dissipation process.

In the special case where $\phi = 0$, the ellipse in Fig. 2 will collapse to a straight line. If in addition $|Q_1| < fP_1$, the contact will remain in a state of stick throughout the loading phase $dP/dt > 0$ and also during the unloading phase, since the system then passes through the same set of equilibrium states as was traversed during loading. In this case there will therefore be no frictional dissipation and one might expect wear and fretting damage to be negligible. By contrast, any non-zero phase lag between the normal and tangential loading is sufficient to ensure that slip occurs at least during the unloading period when $dP/dt < 0$.

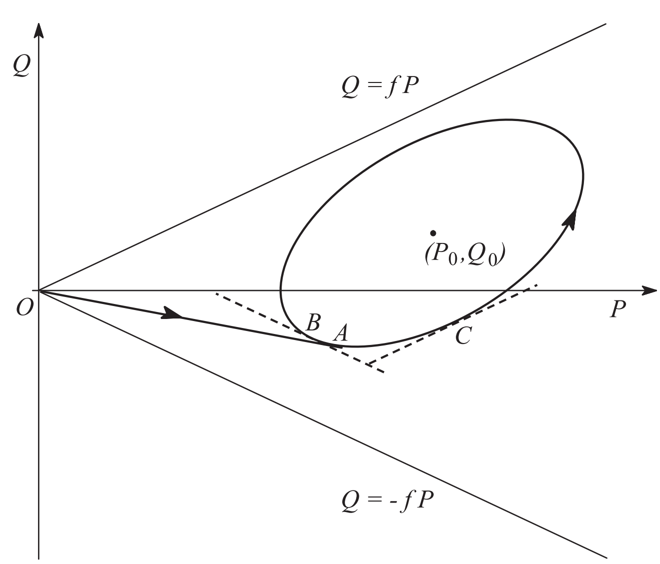

Fig. 2. Periodic loading cycle in PQ-space.

2. Evolution of the traction distribution

Since the materials are similar (or Dundurs constant $\beta = 0$), the normal contact problem is independent of the tangential tractions. In particular, the extent of the contact area $\mathcal{A}$ and the distribution of normal contact pressure $p(x)$ depend only on the instantaneous normal load P, where x is a coordinate defining position in the contact area. We shall indicate this dependence explicitly by defining the functions $p(x,P), \mathcal{A}(P)$. These functions depend only on the profiles of the contacting bodies and the composite elastic modulus and can be determined by various classical methods, including reduction to a Riemann-Hilbert problem and solution by Cauchy integral equations (Muskhelishvili, 1963, Section 119; Hills et al., 1993), Green's functions (Johnson, 1985) or Fourier series (Barber, 2010, Section 12.5). In this paper, we shall assume that the normal contact problem has been solved and that $p(x,P), \mathcal{A}(P)$ are known functions of P. They are non-decreasing functions of P in the sense that $\mathcal{A}(P_1) \in \mathcal{A}(P_2)$ if $P_2 > P_1$ and $p'(x,P) \geqslant 0$ for all x (Barber, 1974), where we introduce the notation

$$p'(x,P) \equiv \frac{\partial p(x,P)}{\partial P}. \tag{2}$$

In terms of these functions, Ciavarella (1998a) and Jäger (1997, 1998) showed that if two solutions of the normal contact problem are identified with the loads P_1, P_2 with $P_2 > P_1$, then the *tangential* traction distribution

$$q(x) = fp(x,P_2) - fp(x,P_1), \tag{3}$$

satisfies the conditions for slip in $x \in \mathcal{A}(P_2) \setminus \mathcal{A}(P_1)$ and for stick (zero relative tangential displacement) in $x \in \mathcal{A}(P_1)$.

The traction distribution (3) clearly satisfies the first of these conditions, since the second term makes no contribution in the slip region $\mathcal{A}(P_2) - \mathcal{A}(P_1)$. The proof that the stick conditions are satisfied in $\mathcal{A}(P_1)$ depends on the fact that the Green's function for normal and tangential loading of the half plane are identical (see for example Johnson (1985) Section 2.2 and 2.3). It follows that the tangential surface displacement due to the traction distribution (3) is equal to the normal surface displacement due to the *difference* between the normal traction distributions $p(x,P_2)$ and $p(x,P_1)$ multiplied by f. This normal traction distribution represents the *change* in normal traction as P is increased from P_1 to P_2 and since the region $\mathcal{A}(P_1)$ is in contact throughout this process, it follows that the normal displacement must be capable of accommodation by a relative rigid body approach and hence must be independent of x. Thus, the tangential surface displacements due to the distribution (3) are also independent of x in $\mathcal{A}(P_1)$ and hence capable of accommodation by a *tangential* relative rigid body displacement. Furthermore, considerations of the asymptotic stress and displacement fields at the transitions between stick and slip (Dundurs and Comninou, 1979) demand that the tangential tractions be bounded at these points, which in combination with the Cauchy integral equation formulation for the tangential problem, ensures that this solution is unique.

2.1. Tangential traction distribution during full stick

In the periodic loading scenario of Fig. 2, the full stick condition $|\Delta Q| < f\Delta P$ is satisfied for all infinitesimal loading increments in the segment BC between the two points B, C where the tangent has slope $\mp f$ respectively implying $dQ/dP = \mp f$. Throughout this segment, the tangential contact problem must be solved incrementally.

We first note that the *change* in normal tractions during a small increase in load ΔP can be written

$$\Delta p(x,P) = p'(x,P)\Delta P. \tag{4}$$

Now during this infinitesimal loading increment, the tangential force must change by

$$\Delta Q = \frac{dQ}{dP}\Delta P$$

and the increment in tangential tractions must ensure that no slip occurs anywhere in the instantaneous contact area $\mathcal{A}(P)$. It is clear from the arguments underlying the Ciavarella-Jäger theorem (or alternatively from the equivalence between the normal and tangential Green's functions for the elastic half plane) that the tangential traction distribution satisfying this condition is proportional to the incremental normal traction distribution and hence

$$\Delta q(x,P) = p'(x,P)\Delta Q = p'(x,P)\frac{dQ}{dP}\Delta P. \tag{5}$$

If the solution of the normal contact problem is known and the loading scenario is defined as a time-directional relation between Q and P, Eq. (5) defines an explicit expression for the increment in tangential tractions during the stick phase. The additional tangential tractions accumulated during the finite stick phase BC (or any other full stick segment, such as OA in Fig. 2) can therefore be written down as the integral

$$q_C(x) - q_B(x) = \int_{P_B}^{P_C} p'(x,P)\frac{dQ}{dP}dP. \tag{6}$$

2.1.1. Partial slip solution

Once we pass the point C in the trajectory of Fig. 2, the condition $d|Q|/dP > f$ is violated and we must anticipate the development of a region of microslip from the edges of the contact area. Suppose the instantaneous load is represented by a point X on the ellipse of Fig. 2 and that the corresponding contact area and normal traction distribution are $\mathcal{A}(P_X), p(x,P_X)$ respectively. We shall demonstrate that the tangential tractions can be expressed as the sum of (i) the tangential tractions $q_Y(x)$ that occurred at some earlier stage Y during the preceding full stick phase such that $P_Y < P_X$ and (ii) a generalized Cattaneo–Mindlin distribution, as in Eq. (3) with $P_2 = P_X$ and $P_1 = P_Y$.

For the first loading cycle, we have chosen an initial loading path that remains in full stick at all times, so our hypothesis is that the tangential tractions at X are defined by

$$q_X(x) = \int_0^{P_Y} p'(x,P)\frac{dQ}{dP}dP + fp(x,P_X) - fp(x,P_Y), \tag{7}$$

where we used (6) to determine $q_Y(x)$. The unknown normal load P_Y is determined from the condition that the resultant tangential force be equal to Q. In fact, this condition gives

$$Q_X = Q_Y + fP_X - fP_Y \quad \text{and hence } Q_Y - fP_Y = Q_X - fP_X \tag{8}$$

and the points X, Y must both lie on the same line $Q - fP = C$ where C is a constant. We can therefore identify Y by drawing such a line on the loading figure and extending it to cut the initial full stick trajectory, as shown in Fig. 3.

The last two terms in (7) comprise a Cattaneo–Mindlin distribution and hence produce no slip in the region $\mathcal{A}(P_Y)$, so the stick boundary condition in this region is satisfied. In the slip region, the tractions are given by $q(x) = fp(x)$ and this is correct provided that the slip velocities have the correct sign in this region. We also require that regions that have once slipped should continue slipping [i.e. that we don't have 'advancing stick zones' as defined by Dundurs and Comninou (1981, 1983)] and hence that $d\mathcal{A}(P_Y)/dt < 0$. Since $\mathcal{A}$ is a non-decreasing function of P, this is equivalent to the condition $dP_Y/dt \leqslant 0$ and it is clear from the geometry of Fig. 3 that this latter condition is satisfied from C all the way around to the point E, where the tangent has slope f, since

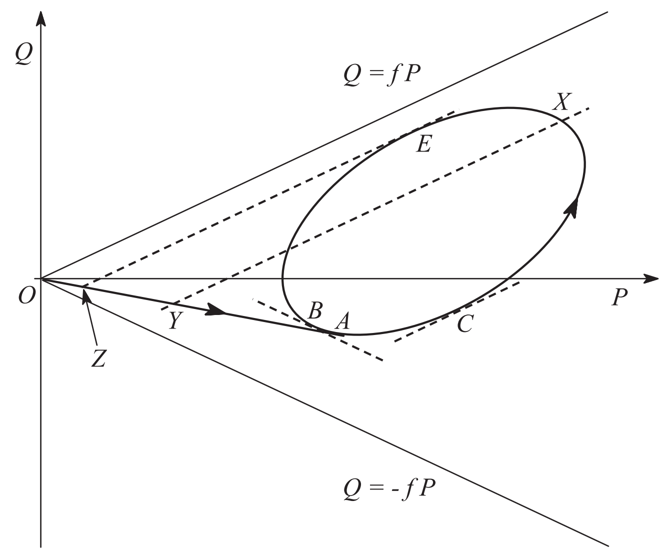

Fig. 3. Geometric construction for determining P_Y in Eq. (7).

as we move X around the loop, the point Y moves continually to the left until X reaches E.

To determine the direction of slip in the postulated slip regions, we first note that at any given stage in the loading process, the incremental tangential tractions during a small loading increment are obtained by differentiating (7) as

$$\Delta q(x) = p'(x, P_Y)\frac{dQ}{dP}\bigg|_{P_Y}\Delta P_Y + fp'(x, P_X)\Delta P_X - fp'(x, P_Y)\Delta P_Y. \tag{9}$$

The second term in this expression is proportional to the normal pressure distribution under a flat punch of planform $\mathcal{A}(P_X)$ and hence produces a uniform tangential displacement throughout this area which can be accommodated by a rigid-body displacement without slip. The remaining two terms can be grouped as

$$p'(x, P_Y)\left(\frac{dQ}{dP}\bigg|_{P_Y} - f\right)\Delta P_Y \tag{10}$$

and, since P_Y lies in a full stick phase where $dQ/dP < f$, the incremental tractions will be positive (and hence produce a positive additional slip) if and only if $\Delta P_Y < 0$, which remains true up to but not beyond the point E. Thus, the slip direction inequality leads to the same criterion as the prohibition on advancing stick. We also note that as we reach the point E the expression (10) goes to zero for all values of x. It follows that the slip rate due to the incremental shear tractions will go to zero throughout the slip region and hence the entire slip region will transition simultaneously at E to a state of stick.

2.2. Reverse slip phase (EB)

Although the system sticks instantaneously at point E, it cannot remain in this state for a finite load interval, because P is decreasing at E and hence the condition $d|Q|/dP < f$ is not satisfied. Instead, starting from E, a region of reverse slip will extend from the edges of the contact area. To analyze this phase of the process, we first note that the tangential tractions at E are defined by

$$q_E(x) = \int_0^{P_Z} p'(x, P)\frac{dQ}{dP}dP + fp(x, P_E) - fp(x, P_Z), \tag{11}$$

from (7) with X replaced by E. Notice that at this point the only 'memory' of the initial loading phase OA is contained in the first and last terms, which are non-zero only in the region $\mathcal{A}(P_Z)$ corresponding to the point Z where the tangent line at E crosses the initial loading line OA. This memory is never erased and hence the final steady state is influenced by the shape of the load trajectory in the range OZ. However, this effect occurs only in the 'residual' tractions inside the permanently stuck zone in the steady state and we shall see later that it has no effect on the frictional slip and hence on dissipation or frictional damage. Furthermore, the total tangential force associated with the first and last terms in (11) is $Q_Z - fP_Z$ and this represents the intercept between the tangent line at E and the Q-axis which is independent of the initial loading path.

At a general point between E and B, the corrective tractions $q_{corr}(x)$ must (i) cancel the term $fp(x,P_E)$ in (11) in the region $\mathcal{A}(P_E) \setminus \mathcal{A}(P)$ (since this region loses contact as P decreases), (ii) satisfy stick conditions in an as yet undetermined stick region $\mathcal{A}_S$, and (iii) convert the tractions to backward slip tractions $-fp(x,P_E)$ in $\mathcal{A}(P) \setminus \mathcal{A}_S$. Notice that if $P_B > P_E$, there will be points in EB for which $\mathcal{A}(P_E) \setminus \mathcal{A}(P)$ is null, in which case only conditions (ii,iii) need to be satisfied.

These conditions are all satisfied by the sum of two Cattaneo–Mindlin distributions defined as

$$q_{corr}(x) = -[fp(x, P_E) - fp(x, P_S)] - [fp(x, P) - fp(x, P_S)], \tag{12}$$

where P_S is a load to be determined that defines the stick region $\mathcal{A}_S$ through the relation $\mathcal{A}_S = \mathcal{A}(P_S)$. Notice that in (12), both terms satisfy the stick condition in $\mathcal{A}(P_S)$, the first term cancels the term $fp(x,P_E)$ in Eq. (11) and the second establishes the required reverse slip tractions $-fp(x,P)$ in $\mathcal{A}(P) \setminus \mathcal{A}(P_S)$. The tractions at a general point between E and B are therefore obtained by adding (11), (12) as

$$q(x) = \int_0^{P_Z} p'(x, P)\frac{dQ}{dP}dP - fp(x, P) + 2fp(x, P_S) - fp(x, P_Z). \tag{13}$$

As before, P_S and hence $\mathcal{A}_S$ is determined from the corresponding tangential equilibrium condition

$$Q = Q_Z - fP + 2fP_S - fP_Z \quad \text{or} \quad P_S = \frac{1}{2}\left(\frac{Q}{f} + P - \frac{Q_Z}{f} + P_Z\right). \tag{14}$$

This load can also be given a simple geometrical description as shown in Fig. 4. If we extend a line of slope $-f$ from the point (P,Q) it will cut the P-axis at G, where $P_G = P + Q/f$. Similarly, the line EZ cuts the P-axis at F, where $P_F = P_Z - Q_Z/f$. It follows that P_S is midway between F and G and hence is defined by the P-coordinate of

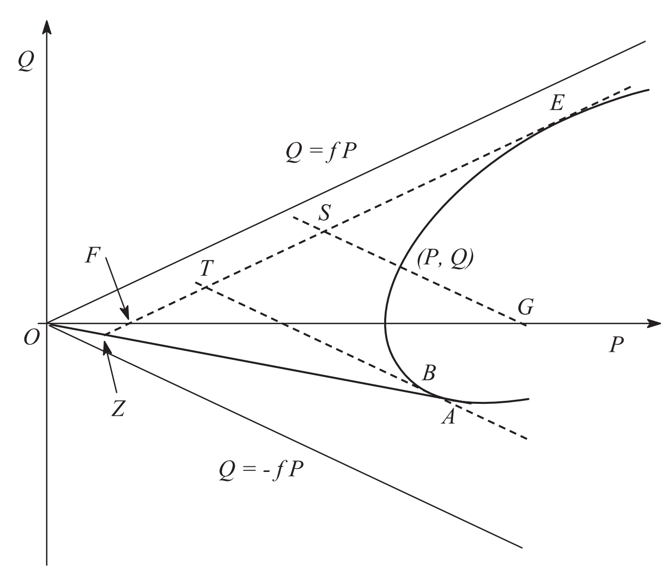

Fig. 4. Geometric construction for determining the stick zone during the reverse slip phase EB.

the intersection point S in Fig. 4. This construction shows that $P > P_S > P_Z$ for all points between E and B, so the tangential displacements locked in the region $\mathcal{A}(P_Z)$ are not changed during this process. Also, P_S and hence the stick zone $\mathcal{A}(P_S)$ continually decrease as we move from E to B, as required if we are to avoid a situation with advancing stick. An argument exactly parallel to that used in the forward slip phase can be used to show that the slip direction in $\mathcal{A}(P) \setminus \mathcal{A}(P_S)$ is consistent with the assumed frictional tractions (13) and once again we find that as we reach the tangent point B, the entire slip zone sticks simultaneously.

At this point, the traction distribution is

$$q_B(x) = \int_0^{P_Z} p'(x,P)\frac{dQ}{dP}dP - fp(x,P_B) + 2fp(x,P_T) - fp(x,P_Z), \qquad (15)$$

where the point T is defined in Fig. 4.

2.3. The second cycle

Between B and C, we shall once again have full stick and the incremental tractions can be obtained by integration as before, giving

$$q(x) = q_B(x) + \int_{P_B}^{P} p'(x,P)\frac{dQ}{dP}dP. \qquad (16)$$

In particular, when we reach C for the second time, we have

$$q_C(x) = \int_0^{P_Z} p'(x,P)\frac{dQ}{dP}dP + \int_{P_B}^{P_C} p'(x,P)\frac{dQ}{dP}dP - fp(x,P_B) + 2fp(x,P_T) - fp(x,P_Z), \qquad (17)$$

from (15), (16).

As in the first cycle, partial slip will start at C and the corrective traction is constructed in the same way as in (7), except that the point Y defining the stick area through $\mathcal{A}(P_Y)$ is now obtained from an intersection with the segment BC rather than with the initial loading line OAC. With this interpretation, the traction distribution in the segment CH in Fig. 5 can be written

$$q(x) = \int_0^{P_Z} p'(x,P)\frac{dQ}{dP}dP + \int_{P_B}^{P_Y} p'(x,P)\frac{dQ}{dP}dP + fp(x,P_X) - fp(x,P_Y) - fp(x,P_B) + 2fp(x,P_T) - fp(x,P_Z). \qquad (18)$$

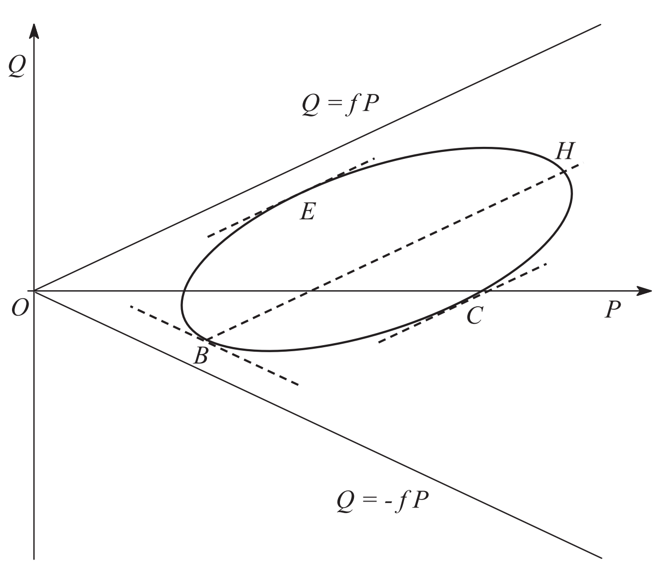

Fig. 5. Tractions accumulated during the reverse slip phase start to be erased after point H.

However, this distribution applies only as long as $\mathcal{A}(P_Y) > \mathcal{A}(P_B)$ and hence $P_Y > P_B$. Once we pass the point H on the second cycle, the slip zone starts to erase the reverse slip tractions locked in at B and a new solution is necessary.

2.4. The phase HE

At the point H, the second integral term in (18) goes to zero, $Y \to B$ and we have

$$q_H(x) = \int_0^{P_Z} p'(x,P)\frac{dQ}{dP}dP + fp(x,P_H) - 2fp(x,P_B) + 2fp(x,P_T) - fp(x,P_Z). \qquad (19)$$

During the phase HE, the contact area is shrinking and we anticipate forward slip will continue near the edges of this region, surrounding a shrinking stick zone. However, the extent of this stick zone cannot now be found by intersecting a line of slope f with the other side of the load loop, because reverse slip was accumulating during the phase EB.

Suppose that the stick area can be expressed as $\mathcal{A}(P_K)$, where P_K is some as yet unknown normal load in the range $P_T < P_K < P_B$. An immediate correction for the reduced size of the contact area can be made by subtracting the Cattaneo–Mindlin distribution $fp(x,P_H) - fp(x,P)$, giving the traction distribution

$$\int_0^{P_Z} p'(x,P)\frac{dQ}{dP}dP + fp(x,P) - 2fp(x,P_B) + 2fp(x,P_T) - fp(x,P_Z).$$

This satisfies the stick condition in $\mathcal{A}(P_K)$ and the forward slip condition in $\mathcal{A}(P) \setminus \mathcal{A}(P_B)$, but the slip condition in $\mathcal{A}(P_B) \setminus \mathcal{A}(P_K)$ is not satisfied because of the term $-2fp(x,P_B)$. However, we can correct this whilst preserving the remaining boundary conditions by adding a further Cattaneo–Mindlin term $2fp(x,P_B) - 2fp(x,P_K)$, giving

$$q(x) = \int_0^{P_Z} p'(x,P)\frac{dQ}{dP}dP + fp(x,P) - 2fp(x,P_K) + 2fp(x,P_T) - fp(x,P_Z) \qquad (20)$$

for the tractions during the phase HE. As before, the unknown force P_K is obtained from the equilibrium condition

$$Q = Q_E + fP - 2fP_K + 2fP_T - fP_Z,$$

or

$$P_K = P_T + \frac{1}{2}\left(P - \frac{Q}{f} - P_Z + \frac{Q_Z}{f}\right) \qquad (21)$$

and the term in brackets must be positive, since the point (P,Q) lies to the right of the tangent line EZ implying that $P - Q/f > P_Z - Q_Z/f$. It follows that $P_K > P_T$ as assumed and also that the two become equal when we reach E. At this point, the forward slip zone has completely erased the tractions accumulated during the reverse slip phase EB and the traction distribution is identical to that at the same point during the first cycle, being given by Eq. (11). It follows that all subsequent loading cycles will follow the same path, so the steady state is reached after one cycle of periodic loading.

2.5. The permanent stick zone

Reviewing the first complete steady-state cycle, starting and ending at point E in Figs. 3–5, we note that the smallest extent of the stick zone is represented by $\mathcal{A}(P_T)$, where the point T is identified in Fig. 4 as the intersection of two tangents of slope $\pm f$ respectively to the load loop. It is clear that this point is independent of the initial loading path OA and hence that the extent of the permanent stick zone is unique. As remarked earlier, the tractions locked into the permanent stick zone *are* influenced by the initial loading path, but the total tangential force transmitted through this zone at any given point in the cycle depends only on the periodic loading cycle. For example, at point E, the tangential tractions are given by Eq. (11) and the sum of the tractions in the permanent stick zone $\mathcal{A}(P_T)$ is

$$\int_{\mathcal{A}(P_T)} q_E(x)dx = Q_Z - fP_Z + f\int_{\mathcal{A}(P_T)} p(x, P_E)dx. \quad (22)$$

The last term in this equation is independent of the initial loading path and we already remarked that the first two terms define the intercept of the tangent line at E with the Q-axis, which clearly depends only on the periodic part of the loading.

We also note that the values of P_T, P_E and hence $\mathcal{A}(P_T)$ would be unchanged if the load loop were to be displaced in the Q-direction whilst preserving its shape and orientation, always assuming that this displacement did not violate the 'no-sliding' condition that the entire loop lie between the bounding lines $Q = \pm fP$. In other words, the extent of the permanent stick zone is independent of Q_0 in (1) subject to this restriction. It also follows that the last term in (22) is independent of Q_0, but the intercept defined by the first two terms will increase by the increment in Q_0. Thus, changing the mean tangential load merely causes the tractions locked into $\mathcal{A}(P_T)$ to increase by an equal amount, whilst the tangential tractions *outside* $\mathcal{A}(P_T)$ are always independent of Q_0, even during periods of stick.

2.6. Energy dissipated in friction

During periods of microslip, work will be done by the tangential tractions in the slip regions, moving through the corresponding slip displacements $u(x)$. The dissipation rate can be written

$$\dot{W} = \int_{\mathcal{A}(P)} q(x)\dot{u}(x)dx, \quad (23)$$

where the dot denotes differentation with respect to time. Now the slip displacement can be written down as an integral of the tangential tractions, and its time derivative $\dot{u}(x)$ depends only on the time derivative of these tractions, which we have established is independent of the initial conditions. It follows that the energy dissipated in friction per cycle depends only on the periodic loading cycle and is also independent of Q_0 in Eq. (1)

2.7. More general load cycles

Most practical loading cycles will result from machine vibration at a dominant frequency and will therefore be quasi-elliptical, as defined by Eq. (1). However, it is clear that the arguments here developed apply to any scenario in which the corresponding loop has a monotonically turning tangent. In fact, more sinuous curves can also be allowed without a change in the procedure as long as they do not precipitate an additional change from forward slip to backward slip, or slip to stick, and this condition will be satisfied as long as the load loop has just four points of tangency with lines of slope $\pm f$.

If there are more than four points of tangency, a complete description of the resulting traction distributions will require a detailed accounting each time such a point is passed, but the arguments of the preceding sections are easily adapted to such cases. For example, in the scenario of Fig. 6, reverse slip will occur between E_1 and B_1, with a maximum slip zone defined by $\mathcal{A}(P_{T_1})$, exactly as in the phase BE of Section 2.1.1. Stick tractions will accumulate between B_1 and C_1, but these and then the reverse slip tractions from E_1B_1 will be erased by a growing forward slip zone between C_1 and G. At this point, all memory of the side loop is erased, the cycle continues in forward slip to E, sticks instantaneously and experiences a growing reverse slip zone to the point B as in Section 2.1.1. The permanent stick zone is given by $\mathcal{A}(P_T)$ as in the simpler scenario and this appears to remain true for all deviations from the quasi-elliptical path. In other words, the permanent stick zone is always defined by the intersection of the two tangents that give the minimum value P_T at the intersection point.

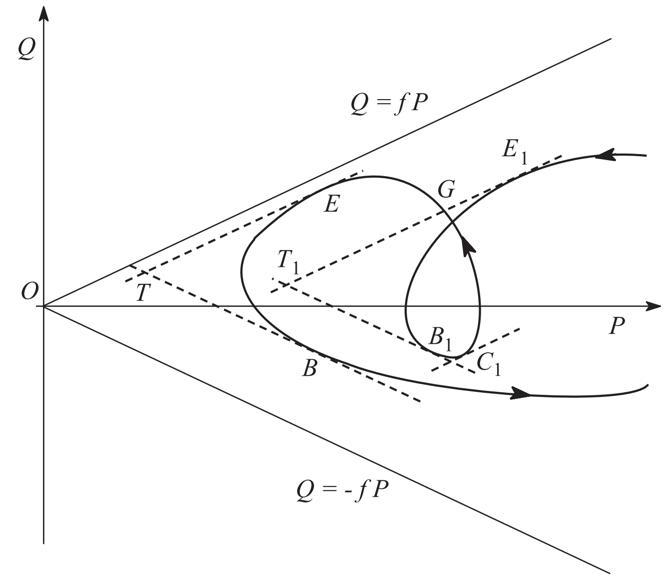

Fig. 6. A more complex load cycle.

3. Three-dimensional problems

The results established in this paper are exact in the context of two-dimensional elasticity, provided that the bodies are sufficiently large to be representable as half planes and Dundurs' constant $\beta = 0$. However, the original papers by Cattaneo (1938) and Mindlin (1949) also discussed the three-dimensional Hertzian contact geometry under the simplifying assumption that the frictional tractions are everywhere aligned with the applied tangential force. This assumption leads to a solution consistent with the Coulomb friction law *if and only if* the resulting slip displacements are also so aligned and for unidirectional loading, this is the case only in the special case where (for similar materials) Poisson's ratio $\nu = 0$. Ciavarella (1998b) showed that in this special case the same superposition can be carried over to general three-dimensional geometries. It follows immediately that all the results of the present paper apply equally to the three-dimensional problem of similar materials, provided that $\nu = 0$ and the tangential load has always the same direction, though possibly alternating in sign. In particular, the contact area function $\mathcal{A}(P)$ will now be a an area (not necessarily connected) in the interfacial plane, and determined from the solution of the normal contact problem.

For all other values of ν, tangential forces produce tangential surface displacements that are inclined to the direction of the force and this can be shown to lead to a violation of the Coulomb law even in the simple Cattaneo–Mindlin case. The extent of this error was examined by Munisamy et al. (1994) using a numerical solution and shown to be generally quite small. Thus, one might expect the present results to apply in some approximate sense to more general three-dimensional problems where $\nu \neq 0$.

4. Conclusions

We have shown that if two two-dimensional elastic bodies are subjected to far-field periodic loading, the following properties of the steady-state frictional behaviour are independent of the initial conditions or an initial transient loading phase, provided the tangential and normal contact problems are uncoupled:-

(i) The extent of the permanent stick zone, comprising points that do not slip or separate during the steady state.
(ii) The tractions at all points outside the permanent stick zone.
(iii) The slip velocities at all such points.
(iv) The energy dissipated per loading cycle due to friction.

This last result can be seen as a generalization of an earlier proof that a frictional Melan's theorem applies to uncoupled frictional elastic systems. We deduce that the effective damping of the contact and the tendency to damage by fretting fatigue should be independent of the way (for example) in which the joint was assembled. We also find that all the above parameters remain unchanged if the loading cycle is modified by the addition of a time-independent tangential force, provided this is not so large as to precipitate a period of gross slip (sliding).

The tractions *inside* the permanent stick zone will generally depend on the initial conditions, but the resultant of these tractions at any given point in the loading cycle will not. These results apply to bodies of completely general profile, subject only to the requirement that they be capable of approximation by half-planes for the purposes of the elastic contact problem. For example, they apply equally to the contact of rough surfaces of specified profile, where contact will occur in a set of microscopic 'actual contact areas'.

The techniques here described also provide a simple algorithm for determining the extent of the permanent stick zone and the traction distributions at all points during the loading cycle.

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