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An incremental formulation for half-plane contact problems subject to varying normal load, shear, and tension

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Abstract: The classical Cauchy integral equation formulation used to solve contact problems capable of representation within a half-plane formulation for elastically similar materials, is developed so that the problem is reducible to an Abel equation for normal load. A fully-adhered solution is then developed permitting the interfacial shearing tractions induced by shear and a difference in remote tensions. This is applied to wedge, Hertz and *flat and rounded* contact problems, and the condtions for sustained adhesion found. The results found have general applicability, but are particularly relevant to the kind of load history experienced in a gas turbine blade root fixing.

Keywords: friction, contact, Hertz, half-plane, proportional loading

1 INTRODUCTION

Contact between elastic bodies, at least one of which is slightly convex, can often be analysed using *half-plane theory*, in which an elasticity formulation appropriate to a half-plane is used to represent each body in the neighbourhood of contact. The starting point for the analysis is to determine the state of stress in a half-plane induced by a line force, as described in [**1**]. From there the surface displacements due to normal and tangential line forces may be found, and superposition used to find the connection between distributed tractions over the contact surface and the corresponding surface displacements. To avoid the unknown rigid-body term, which arises when employing a half-plane, we normally specify the derivatives of the surface displacements and, provided that the contacting bodies are elastically similar, arrive at the following pair of integral equations for the relative normal displacement $v(x) = v_1(x) - v_2(x)$ (see Fig. 1(a)), and relative surface tangential displacement, $u(x) = u_1(x) - u_2(x)$, in terms of the contact pressure, $p(x)$ and shear traction, $q(x)$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\kappa+1}{2\mu\pi}\int_{\text{contact}} \frac{p(\xi)\mathrm{d}\xi}{x-\xi} \tag{1}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\kappa+1}{2\mu\pi}\int_{\text{contact}} \frac{q(\xi)\mathrm{d}\xi}{x-\xi} \tag{2}$$

where μ is the modulus of rigidity, κ Kolosov's constant (equal to $3-4\nu$ in plane strain, where ν is Poisson's ratio). These integrals take a different value if the observation point is within the interval of integration (i.e. the contact) when they assume *Cauchy principal values*, from when the observation point is outside the range of integration (outside the contact), when they are *regular*. Note that the direct traction induces only a change in relative normal displacement gradient and the shear traction a change in relative tangential displacement: the solution is *uncoupled*. The normal contact problem is usually solved first by setting the relative surface

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normal displacement gradient equal in magnitude to the slope of the gap between the bodies before they are pressed together, $g_o'(x)$. The contact pressure is then found by *inversion*. Details of the inversion process were given by Muskhelishvili [**2**] and in a simple form by Hills *et al.* [**3**]. To obtain the contact law, we need to establish equilibrium between the contact pressure and the applied load, P, giving

$$P = \int_{\text{contact}} p(x)\mathrm{d}x \tag{3}$$

Once this has been done, the effect of tangential tractions may be found. Details of the application will dictate whether the tractions are known everywhere (if the contact is sliding) or if the displacements are known everywhere (if the contact is fully adhered) or if the interface has a mixture of slipping and adhered regions (so that a state of partial slip exists).

This method is satisfactory, and is particularly effective when first the normal load is applied and held constant, and then the shearing force is developed, but is not well suited to problems where both the normal and shearing force are varying functions of time. In these cases an alternative method, developed by Mossakovskii [**4**], Spence [**5**], and Jäger [**6**], which builds on the Cauchy integral equations already quoted, and uses them to form an alternate representation of the general contact problem is appropriate. The formulation, which has previously been employed without a remote tension difference between the bodies, leads to Abel-type integral equations instead of the Cauchy ones, allowing a general method that is well suited to frictional problems considered incrementally in terms of normal and shear applied forces along with synchronous bulk tension.

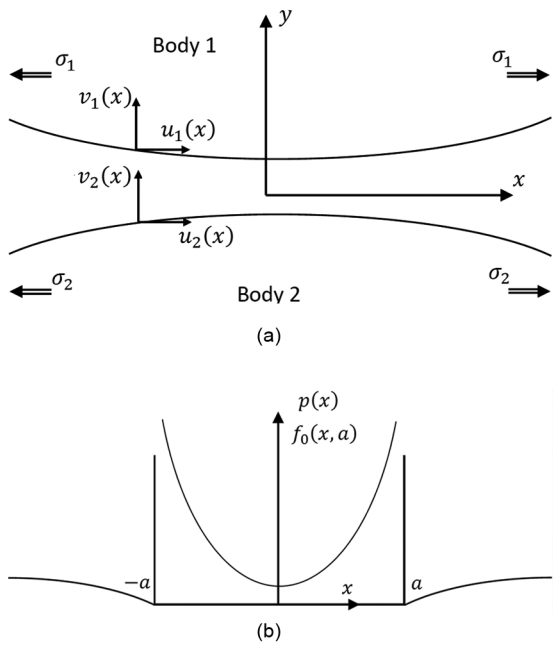

Fig. 1 (a) Definition of the problem geometry. (b) Contact pressure distribution produced by a square ended punch, and corresponding surface displacement

Thinking about the normal contact problem first, the general idea is to replace the smooth front face of the contact defining punch by one consisting of a lot of steps – in the limit infinitesimal steps – and to find the effect of a set of load increments corresponding to each of the steps. As the effect of driving the punch in is simply to increase the displacement imposed on the other body by a constant amount, the key first step is to look at traction distributions which give rise to constant surface displacements.

2 SOLUTION FOR NORMAL LOAD

In preparation for the incremental solution, we note that substituting the normal pressure distribution

$$p(x) = f_0(x,a) \tag{4}$$

where

$$f_0(x,a) = \begin{matrix} \frac{1}{\sqrt{a^2-x^2}}; & -a < x < a \\ 0; & |x| > a \end{matrix} \tag{5}$$

into the first integral (1) and evaluating it produces the combined surface slope

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{(\kappa+1)}{2\mu} h_0(x,a) \tag{6}$$

where

$$h_0(x,a) = \begin{matrix} 0; & -a < x < a \\ -\frac{sgn(x)}{\sqrt{x^2-a^2}}; & |x| > a \end{matrix} \tag{7}$$

The normal surface displacement is shown in Fig. 1(b). Thus, we know the contact pressure which must be applied over the interval $[-a \quad a]$ to produce a constant displacement over the same range, and which corresponds to a total force

$$P = \int_{-a}^{a} p(x)\mathrm{d}x = \pi \tag{8}$$

We are now in a position to define the indentation problem in a different way from that

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conventionally employed. Suppose that we have already pressed the indenter, which has a profile whose slope is defined by $g_0'(x)$, so as to make contact over the interval $(-a \leqslant x \leqslant a)$. Now increase the normal load by a small amount so that, within the existing contact, the bodies suffer a further small constant compression. Additionally, the contact width increases and the compression of material within that small region must be consistent with the form of the contacting profile. In the limit when the increment of load, dP, is very small, and the corresponding change in half-width is ds, we can write the resultant contact pressure, $p(x, a)$, as the sum of a set of load steps and use superposition of the above solution to give

$$p(x, a) = \int_0^a F(s) f_0(x, s) \mathrm{d}s \tag{9}$$

where the corresponding surface relative slope is given by

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{(k+1)}{2\mu} \int_0^s F(s) h_0(x, s) \mathrm{d}s \tag{10}$$

Notice that if $F(s)$ is a continuous function, the singularity in $f_0(x, s)$ will be integrated out, so that equation (9) always defines a distribution that tends to zero at the edges of the contact area $x = \pm a$. In other words, the consistency condition, which has to be imposed separately in the Cauchy equation formulation, is automatically satisfied.

Within the contact region the surface slope must match the slope of the profile, and hence

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -g_0'(x) \qquad -a < x < a \tag{11}$$

Thus

$$\int_0^x \frac{F(s)\mathrm{d}s}{\sqrt{x^2 - s^2}} = \frac{2\mu g_0'(x)}{(k+1)} \qquad 0 < x < a \tag{12}$$

which is an Abel equation for the unknown function $F(s)$ with solution

$$F(s) = \frac{4\mu}{\pi(\kappa+1)} \frac{\mathrm{d}}{\mathrm{d}s} \int_0^s \frac{t g_0'(t)\mathrm{d}t}{\sqrt{s^2 - t^2}} \tag{13}$$

Once $F(s)$ is known for the punch profile under consideration, the pressure distribution may be found from

$$p(x, a) = \begin{cases} \int_{|x|}^a \frac{F(s)\mathrm{d}s}{\sqrt{s^2 - x^2}}; & |x| < a \\ 0 & |x| \geqslant a \end{cases} \tag{14}$$

and, from an imposition overall normal equilibrium, it may be shown that

$$P(a) = \pi \int_0^a F(s)\mathrm{d}s \tag{15}$$

3 TANGENTIAL LOADING

The integral equation connecting the direct traction with the surface relative slope (1) is exactly the same as that connecting the shear traction with the relative surface tangential strain (2). It follows that the results developed previously, in which the contact pressure distribution needed to achieve a constant normal displacement over the interval $[-a \quad a]$, were found to apply also to shear loading of a pair of elastically similar half-planes in glancing contact, free to slide over each other everywhere except the interval $(-a \leqslant x \leqslant a)$ where they are bonded: the application of a remote shearing force, Q, causes a constant surface tangential displacement relative to the interior of each half-plane to be developed, so that the relative tangential surface strain must vanish. The shearing traction present along the interface, $q(x)$ is then given by

$$q(x) = f_0(x) \tag{16}$$

and the difference between the strains induced in the two surfaces $(\varepsilon_{xx}^1(x, 0) - \varepsilon_{xx}^2(x, 0) \equiv \mathrm{d}u/\mathrm{d}x)$ is

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{(\kappa+1)}{2\mu} h_0(x, a) \tag{17}$$

If we apply this result to a contact which is both; (a) advancing, i.e. the normal load is a monotonically increasing function of time, and (b) where the coefficient of friction is sufficient to prevent all slip between surface particles as they are pressed into contact, we see, by analogy with the solution for normal load, that the integral equation specifying the shear traction is given by

$$q(x, a) = \begin{cases} \int_{|x|}^a \frac{G_0(s)\mathrm{d}s}{\sqrt{s^2 - x^2}} & |x| < a \\ 0 & |x| \geqslant a \end{cases} \tag{18}$$

and the corresponding shearing force is given by

$$Q(a) = \pi \int_0^a G_0(s)\mathrm{d}s \tag{19}$$

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We can find $G_0(a)$ by differentiation with respect to a

$$G_0(a) = \frac{1}{\pi}\frac{\mathrm{d}Q}{\mathrm{d}a} \tag{20}$$

4 INFLUENCE OF REMOTE TENSIONS

We now introduce the possibility of surface interfacial shearing tractions arising from a different source, namely the application of remote tensions to the contacting bodies. These frequently arise in practical problems and are important in controlling surface slip and fretting fatigue. A fundamental solution which provides a connexion between surface shear tractions and the corresponding surface strains is needed first. Suppose that we impose a remote tension σ_1 to the upper half-plane and a bulk tension σ_2 to the lower half-plane, while they remain bonded over the interval $(-a \leqslant x \leqslant a)$, Fig. 1(a). The difference in remote surface tangential strain, $\varepsilon(x, t)$, is given by

$$\varepsilon(x, t) = \frac{(\kappa+1)\sigma_0(t)}{8\mu} \tag{21}$$

where $\sigma_0(t) = \sigma_1(t) - \sigma_2(t)$. This, we will show, will demand a constant surface strain within the contact, and associated shearing tractions. We therefore turn, again, to equation (2), but now substitute the shearing traction distribution

$$q(x) = f_1(x) \tag{22}$$

where

$$f_1(x, a) = \begin{matrix} \frac{x}{\sqrt{a^2-x^2}}; & |-a < x < a \\ 0; & ||x| > a \end{matrix} \tag{23}$$

which, because this shear traction is odd in x, produces no resultant force, but gives a relative direct surface strain

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{(\kappa+1)}{2\mu} h_1(x, a) \tag{24}$$

where

$$h_1(x, a) = \begin{matrix} 1; & -a < x < a \\ -\left(\frac{|x|}{\sqrt{x^2-a^2}} - 1\right); & |x| > a \end{matrix} \tag{25}$$

The relative surface strain for a general, advancing, adhered contact problem (which is analogous to the surface slope in the normal contact problem), is given by

$$\varepsilon(x, a) = \frac{(\kappa+1)\sigma_0(t)}{8\mu} + \frac{(k+1)}{2\mu}\int_0^a G_1(s) h_1(x, s)\mathrm{d}s \tag{26}$$

where the first term arises from the strains produced by the remote tensions themselves. The adhesion condition demands that the total strain is constant within the contact, as it develops, and therefore that

$$\frac{\partial \varepsilon(x, a)}{\partial a} = 0 \tag{27}$$

so that

$$G_1(a) = -\frac{1}{4}\frac{\mathrm{d}\sigma_0(a)}{\mathrm{d}a} \tag{28}$$

Again, we note that the loading trajectory specifies $P(t)$ and $\sigma_0(t)$, so that $\sigma_0(P)$, and hence $\sigma_0(a)$, are given. In the general case, as material is laid down both an external shearing force and an external difference in remote tensions may be present, specifying both $G_0(s)$, $G_1(s)$. The complete shearing traction and surface strain distributions may then be found as

$$q(x, a) = \int_{|x|}^a \frac{G_0(s)\mathrm{d}s}{\sqrt{s^2 - x^2}} + \int_{|x|}^a \frac{xG_1(s)\mathrm{d}s}{\sqrt{s^2 - x^2}} \tag{29}$$

$$\begin{aligned} \varepsilon(x, a) = {} & \frac{(\kappa+1)\sigma_0(a)}{8\mu} - \frac{(\kappa+1)\mathrm{sgn}(x)}{2\mu}\int_0^{|x|} \frac{G_0(s)\mathrm{d}s}{\sqrt{x^2 - s^2}} \\ & + \frac{(\kappa+1)}{2\mu}\int_0^a G_1(s)\mathrm{d}s - \frac{(k+1)|x|}{2\mu}\int_0^{|x|} \frac{G_1(s)\mathrm{d}s}{\sqrt{x^2 - s^2}} \end{aligned} \tag{30}$$

To summarize, the solution to the problem, in the sense of the surface tractions and direct strain, are fully specified by the three functions $F(s)$, $G_0(s)$, $G_1(s)$. The first of these is, in turn, defined by the geometry of the indenter (equation (1)), while the other two are defined by the loading trajectory – the relative rates at which the shearing force and differential bulk tension are developed. To illustrate the method, three example geometries, the Hertz problem, a *flat punch with rounded edges* punch, and a shallow symmetric wedge, are evaluated explicitly in the Appendix.

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5 PROPORTIONAL LOADING AND THE CONDITION FOR FULL-STICK

Suppose that a pair of contacting bodies is subject to a set of loads $P(t)$, $Q(t)$, $\sigma_0(t)$. We consider now what happens during a time step Δt, when there are corresponding changes in the load ΔP $(>0), \Delta Q, \Delta\sigma_0$. The necessary and sufficient condition for complete adhesion is that $|\Delta q| < f\Delta p$ at all points within the contact. The above formulation lends itself extremely well to a consideration of this condition, because we see immediately (from equations (14) and (15)) that

$$\Delta p(x) = \frac{\Delta P}{\pi\sqrt{a^2 - x^2}} \tag{31}$$

where a is the current contact length, and this result is independent of the punch geometry (i.e. of $a(P)$). Similarly, we have also shown that a change in the shearing force and relative remote tensions produces a change in shearing traction given by

$$\Delta q(x) = \frac{\Delta Q}{\pi\sqrt{a^2 - x^2}} + \frac{\Delta\sigma_0 a}{4\sqrt{a^2 - x^2}} \tag{32}$$

from equations (28) and (29). The ratio of $\Delta|q(x)|\Delta p(x)$ will be constant across the interface, so the entire contact will either be fully stuck or sliding, partial slip cannot occur. The condition for no slip at any point within the contact is clearly

$$\frac{|\Delta Q|}{\Delta P} + \frac{\pi\Delta\sigma_0 a}{4\Delta P} < f \tag{33}$$

This result is independent of the form of the indenter generating the contact.

We now consider what happens when a contact is gradually loaded so that the elements of loading are proportional

$$\frac{\mathrm{d}P}{P} = \frac{\mathrm{d}Q}{Q} = \frac{\mathrm{d}\sigma_0}{\sigma_0} \tag{34}$$

and equation (33) becomes

$$\frac{|Q|}{P} + \frac{\pi\sigma_0 a}{4P} < f \tag{35}$$

where P, Q, σ_0 are their current values, and a is the contact width corresponding to P for the particular contact geometry being considered.

5.1 Examples

The result found is geometry independent, but in practice P and a are connected by the contact law. We could, for the case of a Hertzian contact (see the Appendix), just eliminate a to give

$$\frac{|Q|}{P} + \frac{\pi\sigma_0}{4}\sqrt{\frac{R(1+\kappa)}{P\mu\pi}} < f \tag{36}$$

or the solution could be written in a slightly more familiar form by denoting $p(0)$ by p_0, so that we may specify the loading trajectory by two dimensionless parameters m, n where

$$Q = mP \tag{37}$$

$$\sigma_0 = np_0 \tag{38}$$

and the requirement for full adhesion is that

$$|m| + \frac{n}{2} < f \tag{39}$$

This simple result, together with the loading regimes which give rise to partial slip and sliding, is shown in Fig. 2. The method is also applied to a punch having the form of a flat face (of half-length b) with rounded edges. This configuration is of particular interest in the gas turbine application, where it approximates the geometry of a fanblade dovetail root. The condition for full adhesion, together with the shear traction distribution locked in as material is laid down is given in the Appendix, and the condition for full adhesion portrayed in Fig. 2 as a function of b/a, maintaining the same definitions of m, n used for the Hertz case. An example shear traction distribution for this problem, where the load has been increased until $b/a = 0.8$ is shown in Fig. 3, for $n/m = 0$ (no bulk load), 0.2, 0.5.

In the case of a wedge of external (shallow) angle ϕ, the requirement for enduring adhesion, making use of the contact law, is that

$$\frac{|Q|}{P} + \pi P\sigma_0\left(\frac{(1+\kappa)}{8\mu\phi}\right)^2 < f$$

6 CONCLUSION

A method of deriving the contact law and traction distribution for general half-plane contact problems has been developed, from the usual Cauchy relations. The solution, as written, applies to symmetrical contacts and where the contacting bodies have

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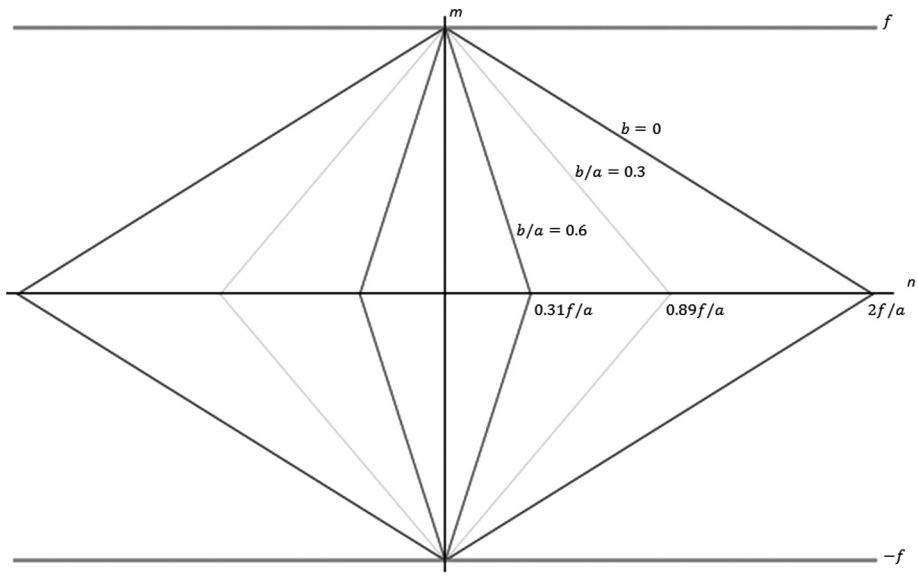

Fig. 2. Resulting state of stick and slip for a flat and rounded punch, including the Hertz case ($b = 0$)

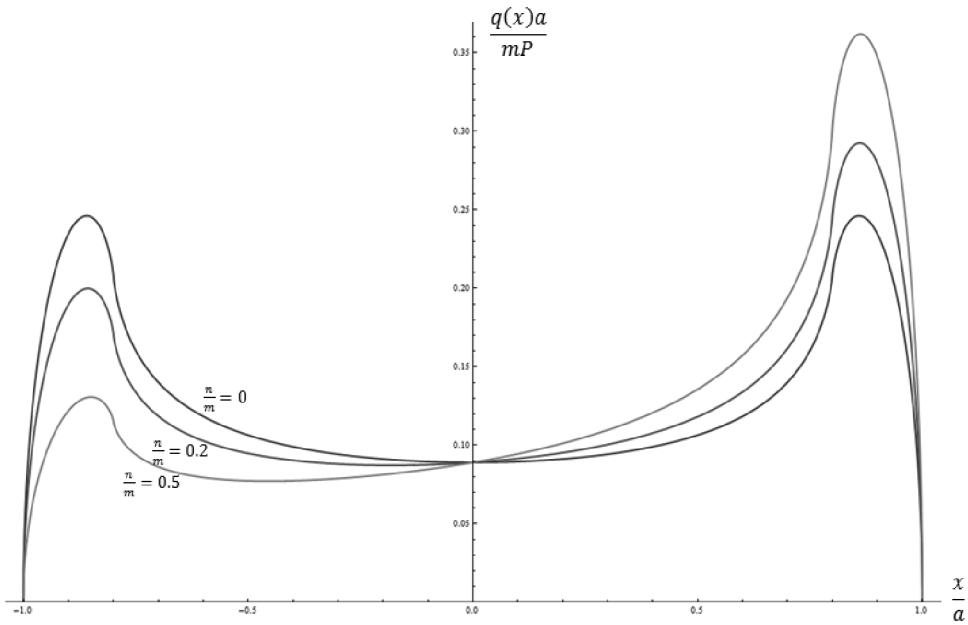

Fig. 3 Shear traction distribution for a flat and rounded punch, $b/a = 0.8$

the same elastic constants, so that they are uncoupled. The normal contact solution is therefore independent of the shear traction distribution, and therefore of the coefficient of friction, whereas the shear traction distribution derived applies whenever the contact is fully adhered. Excitations of shearing traction by both an external shear force and differential tensions in the contacting surfaces are considered. The conditions for complete adhesion of the contact are then derived, and example application of the results to Hertzian and other contacting geometries are carried out. Because the formulation is developed in an incremental fashion it is particularly well suited to application of many frictional contact problems having a complex loading trajectory (not just those where full adhesion is ensured) and many of the intermediate results found are of general interest and applicability.

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APPENDIX

Three Example Geometries

1. The Hertz problem, for a cylinder of radius R, where the slope of the indenter varies linearly with position

$$g_0'(x) = \frac{x}{R} \tag{40}$$

so that

$$F(s) = \frac{2\mu s}{(\kappa+1)R} \tag{41}$$

and

$$p(x,a) = \frac{2\mu\sqrt{a^2-x^2}}{(\kappa+1)R} \tag{42}$$

$$P = \frac{\pi\mu a^2}{(\kappa+1)R} \tag{43}$$

2. Flat and rounded punch of flat half-length b, with edge radii R

$$g_0'(x) = 0 \qquad |x| < b \tag{44}$$

$$= \frac{|x|-b}{R} sgn(x) \qquad |x| > b \tag{45}$$

so that

$$F(s) = 0 \qquad s < b \tag{46}$$

$$F(s) = \frac{4\mu}{R(1+\kappa)} s\cos^{-1}\frac{b}{s} \qquad |s| > b \tag{47}$$

and

$$p(x,a) = \frac{4\mu}{R(\kappa+1)}(I_1) \tag{48}$$

$$P = \frac{2\mu\pi}{R(1+\kappa)}\left(a^2\cos^{-1}\frac{b}{a} - b\sqrt{a^2-b^2}\right) \tag{49}$$

where

$$\begin{aligned} I_1 \equiv & \int_x^a \frac{s\cos^{-1}\frac{b}{s}\,\mathrm{d}s}{\sqrt{s^2-x^2}} \\ & \sqrt{a^2-x^2}\cos^{-1}\tfrac{b}{a} - x\log a - \tfrac{1}{2}(b+x)\log\left(x^2-b^2\right) \\ = & +\tfrac{b}{2}\log\left(2a^2-b^2-x^2-2\sqrt{(a^2-b^2)(a^2-x^2)}\right) \quad -a<x<a \\ & +\tfrac{x}{2}\log\left(a^2\left(b^2+x^2\right)+2bx\sqrt{(a^2-b^2)(a^2-x^2)}-bx\right) \end{aligned} \tag{50}$$

and the condition for partial slip is

$$m + \frac{\pi n}{4}p^* < f \tag{51}$$

where p^* is defined as

$$p^* = \frac{a^2\cos^{-1}\frac{b}{a} + \frac{ab}{2}\log\left(2\frac{a^2}{b^2} - 1 - \frac{2a}{b^2}\sqrt{a^2-b^2}\right)}{a^2\cos^{-1}\frac{b}{a} - b\sqrt{a^2-b^2}} \tag{52}$$

The shear traction distribution requires $G_1(s)$ and $G_2(s)$, which are given by

$$G_0(s) = \frac{1}{\pi}\frac{\mathrm{d}Q}{\mathrm{d}s} = \frac{m}{\pi}\frac{\mathrm{d}P}{\mathrm{d}s} = \frac{4m\mu}{R(1+\kappa)} s\cos^{-1}\frac{b}{s} \tag{53}$$

$$\begin{aligned} G_1(s) &= -\frac{1}{4}\frac{\mathrm{d}\sigma_0(s)}{\mathrm{d}s} = -\frac{n}{4}\frac{\mathrm{d}p_0}{\mathrm{d}s} \\ &= -\frac{n\mu}{R(1+\kappa)}\cos^{-1}\frac{b}{s} \end{aligned} \tag{54}$$

and the shear traction distribution is

$$q(x,a) = \frac{4\mu}{R(1+\kappa)} \left(\frac{m}{\pi}\int_{|x|}^a \frac{s\cos^{-1}\frac{b}{s}}{\sqrt{s^2-x^2}}\mathrm{d}s - \frac{nx}{4}\int_{|x|}^a \frac{\cos^{-1}\frac{b}{s}}{\sqrt{s^2-x^2}}\mathrm{d}s\right) \tag{55}$$

The result of the first integral is given by equation (50), while the second must be evaluated numeri-cally.

3. The wedge problem, where the slope of the indenter is constant and of external (shallow) angle ϕ

$$g_0'(x) = \phi sgn(x) \tag{56}$$

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so that

$$F(s) = \frac{4\mu\phi}{\pi(1+\kappa)} \tag{57}$$

and

$$p(x, a) = \frac{4\mu\phi}{\pi(1+\kappa)} \cosh^{-1} \frac{a}{|x|} \tag{58}$$

$$P = \frac{4\mu\phi a}{(1+\kappa)} \tag{59}$$

$$a = \frac{P(1+\kappa)}{4\mu\phi} \tag{60}$$

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