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Thermoelastic Contact Problems for the Layer

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Introduction

Thermoelastic contact problems involving layers arise in a variety of engineering applications, including disk brakes and clutches (Lee and Barber, 1993) and unidirectional solidification of castings (Richmond et al., 1990). A typical problem might involve the (frictionless) unilateral contact relations

$$g(x) = 0;\ p(x) > 0 \quad \text{in contact regions,} \tag{1}$$

$$p(x) = 0;\ g(x) > 0 \quad \text{in separation regions,} \tag{2}$$

where $p(x)$ is the contact pressure, $g(x)$ is the gap between the surfaces, and the inequalities serve to determine the extent of the contact region. Notice that $g(x)$ is determined by the displacement component $u_y(x)$ at the contact surface and a function describing the shape and rigid-body displacement of the other contacting body.

In order to enforce these conditions, we need to know the relations between $p(x)$, $u_y(x)$ and the (presumably known) temperature profile $T(x, y)$ in the layer. In some special cases, these results can be obtained without determining the entire thermoelastic stress and displacement fields. For example, if the temperature field is in the steady state and the boundaries are traction-free ($p(x) = 0$), Dundurs (1974) has shown that

$$\frac{\partial^2 u_y}{\partial x^2} = -\beta \frac{\partial T}{\partial y} = \frac{\beta}{K} q_y, \tag{3}$$

where q_y is the local heat flux in the y-direction,

$$\beta = \alpha \quad \text{in plane stress,} \tag{4}$$

$$= \alpha(1 + \nu) \quad \text{in plane strain} \tag{5}$$

and α, ν, K are, respectively, the coefficient of thermal expansion, Poisson's ratio, and thermal conductivity for the material. Imposition of this condition at the contact surface provides a relation between the heat input (or temperature gradient) at the boundary and the corresponding distorted shape of the surface. Applications to contact problems are discussed by Barber (1980).

The Half-Plane

Results not restricted to steady-state temperature fields have been obtained for the case of the half-plane $y > 0$, which can be regarded as the limit of a thick layer. Ruiz Ayala et al. (1996) showed that if the half-plane is pressed against a frictionless rigid plane with a sufficient mean pressure to maintain full contact (and hence $u_y(x) = 0$) and if the temperature in the half-plane has the form

$$T(x, y) = T_m(y) \cos(mx), \tag{6}$$

the corresponding perturbation in contact pressure due to the temperature field will be

$$p(x) = p_m \cos(mx), \tag{7}$$

where

$$p_m = \frac{8\beta\mu m}{(\kappa + 1)} \int_0^\infty e^{-ms} T_m(s) ds, \tag{8}$$

μ is the modulus of rigidity, and

$$\kappa = \left(\frac{3 - \nu}{1 + \nu}\right) \quad \text{in plane stress,} \tag{9}$$

$$= (3 - 4\nu) \quad \text{in plane strain.} \tag{10}$$

The isothermal half-plane subjected to a contact pressure of the form (7) develops a surface waviness described by

$$u_y(x) = u_m \cos(mx), \tag{11}$$

where

$$u_m = \frac{(\kappa + 1)p_m}{4\mu m} \tag{12}$$

(see, for example, Johnson, 1985, Section 13.2).

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Superposing these results, we find the following general relation between p_m, u_m, $T_m(y)$ for the half-plane:

$$p_m = \frac{8\beta\mu m}{(\kappa+1)} \int_0^\infty e^{-ms} T_m(s) ds + \frac{4\mu m u_m}{(\kappa+1)}. \quad (13)$$

Generalization to an arbitrary x-variation of these quantities is achieved by writing them as Fourier transforms, i.e.,

$$p(x) = \int_{-\infty}^{\infty} e^{imx} p(m) dm; \; u_y(x) = \int_{-\infty}^{\infty} e^{imx} u(m) dm;$$

$$T(x, y) = \int_{-\infty}^{\infty} e^{imx} T(m, y) dm, \quad (14)$$

where

$$p(m) = \frac{8\beta\mu m}{(\kappa+1)} \int_0^\infty e^{-ms} T(m, s) ds + \frac{4\mu m u(m)}{(\kappa+1)}. \quad (15)$$

The purpose of the present note is to establish a similar result for the case of the layer of finite thickness $2h$.

The Thermoelastic Solution

Following Ruiz Ayala et al. (1996), we first consider the layer $-h < y < h$ with the temperature field of Eq. (6) and confined between frictionless rigid planes, so that

$$u_y = 0; \; \sigma_{yx} = 0; \; y = \pm h. \quad (16)$$

A particular solution to the thermoelastic equations can be constructed in the form

$$2\mu \mathbf{u} = \nabla \phi \quad (17)$$

(Westergaard, 1964), where the scalar potential function ϕ satisfies the equation

$$\nabla^2 \phi = \frac{8\beta\mu T}{(\kappa+1)}. \quad (18)$$

In view of the form of the temperature distribution (6), it is appropriate to consider functions of the form

$$\phi(x, y) = \frac{8\beta\mu}{(\kappa+1)} f(y) \cos(mx), \quad (19)$$

in which case the governing Eq. (18) reduces to the ordinary differential equation

$$\frac{d^2 f}{dy^2} - m^2 f = T_m(y). \quad (20)$$

The left-hand side of this equation can be factorized into two linear operators. Appropriate integrating factors then permit the equation to be written

$$e^{my} \frac{d}{dy} \left\{ e^{-2my} \frac{d}{dy} (e^{my} f) \right\} = T_m(y), \quad (21)$$

the solution of which is

$$f(y) = e^{-my} \int_{-h}^{y} \int_{-h}^{z} e^{(2mz - ms)} T_m(s) ds dz + A \cosh(my) + B \sinh(my), \quad (22)$$

where A, B are arbitrary constants of integration.

The integral term in (22) can be simplified by changing the order of integration and performing the inner integral, with the result

$$f(y) = \frac{1}{m} \int_{-h}^{y} \sinh(m(y-s)) T_m(s) ds + A \cosh(my) + B \sinh(my). \quad (23)$$

The stress components associated with the solution of Eqs. (17), (18) are

$$\sigma_{xx} = -\frac{\partial^2 \phi}{\partial y^2}; \; \sigma_{xy} = \frac{\partial^2 \phi}{\partial x \partial y}; \; \sigma_{yy} = -\frac{\partial^2 \phi}{\partial x^2}. \quad (24)$$

It follows that all the four boundary conditions (16) can be satisfied by imposing the two conditions

$$f'(-h) = 0; \; f'(h) = 0, \quad (25)$$

which in turn can be satisfied with an appropriate choice of the constants A, B. Thus, with these particular boundary conditions, it is not necessary to supplement the particular solution with a homogeneous solution.

We are interested only in the contact pressures

$$p(\pm h) = -\sigma_{yy}(x, \pm h) = -\frac{8\beta\mu m^2}{(\kappa+1)} f(\pm h) \cos(mx). \quad (26)$$

Using (25) to find A, B and substituting into (23), we obtain, after some routine manipulations,

$$f(\pm h) = -\frac{1}{m \sinh(2mh)} \int_{-h}^{h} \cosh(m(s \pm h)) T_m(s) ds \quad (27)$$

and hence

$$p_m(\pm h) = \frac{8\beta\mu m}{(\kappa+1)\sinh(2mh)} \times \int_{-h}^{h} \cosh(m(s \pm h)) T_m(s) ds. \quad (28)$$

In the limit $mh \gg 1$ it can be shown that these results tend to Eq. (8), with an appropriate change of coordinate system.

The Isothermal Solution

To introduce the effect of a nonzero gap function $g(x)$, we need the layer equivalent of the isothermal solution of Eqs. (11), (12)—i.e., the contact pressure distribution needed to satisfy the boundary conditions

$$\sigma_{yx}|_{y=h} = \sigma_{yx}|_{y=-h} = 0; \; u_y|_{y=h} = u_{1m} \cos(mx); \; u_y|_{y=-h} = u_{2m} \cos(mx). \quad (29)$$

This is a routine boundary value problem in elasticity and the details of the solution will be omitted here for brevity. A convenient solution method is to use solutions A and D of Green and Zerna (1954), with harmonic potential functions of the form (sinh (my), cosh (my)) cos (mx).

We find that

$$p(h) \equiv -\sigma_{yy}(h) = p_{1m} \cos(mx); \; p(-h) \equiv -\sigma_{yy}(-h) = p_{2m} \cos(mx), \quad (30)$$

where

$$p_{1m} = \frac{2\mu m}{(\kappa+1)} (u_{2m} f_2(mh) - u_{1m} f_1(mh)), \quad (31)$$

$$p_{2m} = \frac{2\mu m}{(\kappa+1)} (u_{2m} f_1(mh) - u_{1m} f_2(mh)) \tag{32}$$

and

$$f_1(s) = \frac{2\cosh^3(s)\sinh(s) - \cosh(s)\sinh(s) + s}{\sinh^2(s)\cosh^2(s)}, \tag{33}$$

$$f_2(s) = \frac{\cosh(s)\sinh(s) + 2s\cosh^2(s) - s}{\sinh^2(s)\cosh^2(s)}. \tag{34}$$

Complete Solution

Superposing the isothermal and thermoelastic solutions, we obtain the following relations between the contact pressure and displacement amplitudes p_{1m}, p_{2m}, u_{1m}, u_{2m} and the temperature function $T_m(y)$:

$$p_{1m} = \frac{2\mu m}{(\kappa+1)} \left(\frac{4\beta}{\sinh(2mh)} \int_{-h}^{h} \cosh(m(s+h)) T_m(s) ds + u_{2m} f_2(mh) - u_{1m} f_1(mh) \right), \tag{35}$$

$$p_{2m} = \frac{2\mu m}{(\kappa+1)} \left(\frac{4\beta}{\sinh(2mh)} \int_{-h}^{h} \cosh(m(s-h)) T_m(s) ds + u_{2m} f_1(mh) - u_{1m} f_2(mh) \right). \tag{36}$$

When using these results in connection with conditions (1), (2), note that a positive displacement $u_y(x, -h)$ increases the gap $g_2(x)$ at $y = -h$, but a positive value of $u_y(x, h)$ *reduces* $g_1(x)$ at $y = h$. Extension to more general distributions by Fourier transformation is routine, as in Eqs. (14), (15) above.

Alternatively, Eqs. (35), (36) can be inverted to express the displacements in terms of the pressures, i.e.,

$$u_{1m} = \frac{(\kappa+1)(p_{2m} f_4(mh) - p_{1m} f_3(mh))}{8\mu m} + \frac{\beta}{\sinh(2mh)} \int_{-h}^{h} [f_3(mh)\cosh(m(s+h)) - f_4(mh)\cosh(m(s-h))] T_m(s) ds, \tag{37}$$

$$u_{2m} = \frac{(\kappa+1)(p_{2m} f_3(mh) - p_{1m} f_4(mh))}{8\mu m} + \frac{\beta}{\sinh(2mh)} \int_{-h}^{h} [f_4(mh)\cosh(m(s+h)) - f_3(mh)\cosh(m(s-h))] T_m(s) ds, \tag{38}$$

where

$$f_3(s) = \frac{2\sinh(s)\cosh^3(s) - \sinh(s)\cosh(s) + s}{\sinh^2(s)\cosh^2(s) - s^2}, \tag{39}$$

$$f_4(s) = \frac{2s\cosh^2(s) + \sinh(s)\cosh(s) - s}{\sinh^2(s)\cosh^2(s) - s^2}. \tag{40}$$

Conclusions

Equations (35), (36) or (37), (38) enable us to formulate the general thermoelastic contact problem for the layer in terms of boundary values only. By setting $p_{1m} = p_{2m} = 0$ in Eqs. (37), (38), we can also determine the distorted shape of the traction-free layer due to an arbitrary sinusoidal temperature distribution.

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On the Application of Potential Theory in Piezoelasticity

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Introduction

A recent state-of-the-art survey by Rao and Sunar (1994) has revealed the wide and important applications of piezoelectric materials (PZMs) in many branches of science and technology. A great number of theoretical works thus have appeared, concerned with various aspects of piezoelectric problems, such as vibrations of plates and shells (Tiersten, 1969; Ding et al., 1997b), inhomogeneities and cracks (Deeg, 1980; Pak, 1990), and general solutions and Green's functions (Ding et al., 1996, 1997a, c), among others.

Though the potential theory has been shown to be very useful in analyzing mixed and mixed-mixed boundary value problems in elasticity (Fabrikant, 1989, 1991), it seems no one has noticed its application in piezoelasticity to date. In this note, we therefore intend to show how to generalize the potential theory to analyze mixed boundary value problems in piezoelasticity. To this end, the problem of a piezoelectric half-space subjected to mixed boundary conditions on its surface is analyzed. Further developments can be expected based on the generalized potential theory method proposed here as well as the elegant results of Fabrikant (1989, 1991).

Basic Formulations

In Cartesian coordinates (with the z-axis being normal to the plane of isotropy), the linear constitutive relations of a transversely isotropic piezoelectric medium (class 6 mm) are (Tiersten, 1969)

$$\sigma_x = c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z},$$

$$\sigma_y = c_{12}\frac{\partial u}{\partial x} + c_{11}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z},$$

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