On wedged configurations with Coulomb friction

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Abstract. If the coefficient of friction is sufficiently large, elastic contact problems can exhibit 'wedged' solutions in which the body remains in a state of stress in the absence of applied loads. In this paper, we demonstrate that the critical coefficient of friction for wedging to occur is also the lowest real eigenvalue of a certain non-linear eigenvalue problem. Possible strategies for solving this eigenvalue problem are discussed.

1 Introduction

The elementary Coulomb friction law is a fruitful source of challenging mathematical questions in contact mechanics. Existence theorems for continuous static and quasi-static elastic contact problems involving Coulomb friction can be proved only subject to the condition that the coefficient of friction be sufficiently small (see [1, 2, 3, 4] for the static case and [5, 6] for the quasi-static case). Existence and uniqueness results for small friction coefficients were reported for finite element discretizations of the problem (with or without regularization techniques of the frictional contact conditions) by Oden and Pires [7], Haslinger [8], and Kikuchi and Oden [9]. The critical coefficient of friction below which the solution is unique depends upon the geometry and elastic properties of the system and no technique is presently available for determining it.

In this paper we shall be concerned with a particular category of non-unique solution associated with the phenomenon of *wedging*, in which the contacting bodies may remain in a self-sustaining stressed state even when the externally applied forces are removed. This is of concern in automated assembly operations [10, 11] where design requirements often necessitate close tolerances for the mating components, but slight misalignment in the assembly operation can then lead to an incorrectly assembled wedged state. In particular, we shall discuss possible methods of determining the critical coefficient of friction above which wedged solutions can exist.

2 Klarbring's model

Early insight into the physical basis behind the non-uniqueness of Coulomb friction solutions was provided by Klarbring [12, 13], who investigated the behavior of an elastically supported block in frictional contact with a rigid plane. He considered the case where an external force **F** increases monotonically with time, whilst maintaining the same direction, so the incremental quasi-static solution should be identical with the static solution. By assuming each possible state (stick, forward slip, backward slip, separation) in turn and solving the resulting linear equations of motion, he was able to identify a critical coefficient of friction f_c such that only one state was possible for a given load direction for $f < f_c$. For $f \geq f_c$ three solutions are obtained, corresponding to stick, separation and one direction of slip respectively [14].

Elementary calculations show that wedged states are also possible for Klarbring's model if and only if $f \geq f_c$ [15]. Thus, for Klarbring's model, wedging is always possible if the monotonic quasi-static solution is non-unique. Suppose now that the coefficient of friction is exactly equal to f_c. The multiple wedged states will still be solutions for the unloaded system, but they will now correspond to states of 'impending slip'. This suggests an alternative strategy for determining the critical coefficient of friction for wedging. We assume slip throughout the contact area and determine the conditions where non-trivial solutions exist with no external loading.

3 The eigenvalue problem

This method of determining the critical friction coefficient is actually a degenerate case of an eigenvalue problem defined by Hassani *et al.* [16] and Hild [17]. These authors demonstrated that discrete [16] and continuous [17] two-dimensional systems can posess an infinity of quasi-static solutions all involving slip for certain particular values of the coefficient of friction and that these critical coefficients can be determined by solving a linear eigenvalue problem.

Fig.2 shows a fairly general two-dimensional system consisting of an elastic body Ω, with boundary $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$, which is supported (displacement $\mathbf{u} = 0$) in Γ_D, loaded by prescribed tractions **F** in Γ_N, and which makes frictional contact with a rigid plane surface in Γ_C. We define the vector of normal contact pressures as **P** and the corresponding vector of tangential nodal forces (positive when acting to the right) as **Q**. The corresponding normal and tangential displacements in Γ_C are written **u**, **v** respectively and the vector **w** represents all the remaining nodal displacements in the discretization.

If there is sliding friction in the same direction throughout Γ_C, we must have

$$\mathbf{u} = \mathbf{0}\ ; \qquad \mathbf{Q} = f\mathbf{P}\ , \tag{1}$$

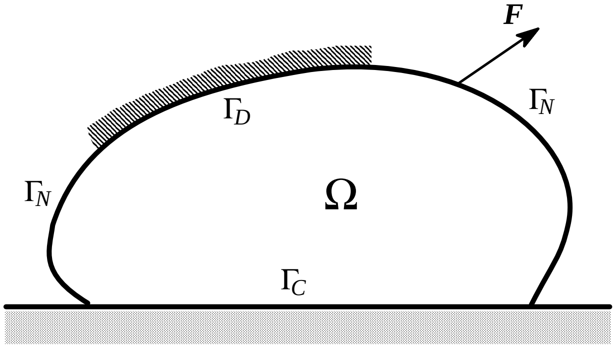

Fig. 1. Two-dimensional elastic contact problem

where f may be of either sign depending on the direction of slip. Suppose there exist two quasi-static solutions to this problem for given applied loads $\mathbf{F}$ and identify the deformation and stress fields by $\mathbf{S}_1$, $\mathbf{S}_2$, respectively. The boundary conditions (1) are linear and hence we can use linear superposition to show that the field $\mathbf{S} = \mathbf{S}_1 - \mathbf{S}_2$ also satisfies (1) and corresponds to zero applied tractions $\mathbf{F} = \mathbf{0}$. Thus, the existence of distinct multiple solutions involving slip also implies the existence of non-trivial solutions to the corresponding homogeneous (unloaded) problem.

With a suitable finite element discretization, there will be a finite number of nodes N in Γ_C and if the external loads $\mathbf{F} = 0$, there will be a linear relation between the contact tractions and displacements which can be written

$$\mathbf{K}\left\{\begin{matrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\end{matrix}\right\} = \begin{bmatrix}\mathbf{K}_{nn} & \mathbf{K}_{nt} & \mathbf{K}_{ni}\\ \mathbf{K}_{nt} & \mathbf{K}_{tt} & \mathbf{K}_{ti}\\ \mathbf{K}_{ni} & \mathbf{K}_{ti} & \mathbf{K}_{ii}\end{bmatrix}\left\{\begin{matrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\end{matrix}\right\} = \left\{\begin{matrix}\mathbf{P}\\ \mathbf{Q}\\ \mathbf{0}\end{matrix}\right\}, \tag{2}$$

or

$$\mathbf{C}\left\{\begin{matrix}\mathbf{P}\\ \mathbf{Q}\\ \mathbf{0}\end{matrix}\right\} = \begin{bmatrix}\mathbf{C}_{nn} & \mathbf{C}_{nt} & \mathbf{C}_{ni}\\ \mathbf{C}_{nt} & \mathbf{C}_{tt} & \mathbf{C}_{ti}\\ \mathbf{C}_{ni} & \mathbf{C}_{ti} & \mathbf{C}_{ii}\end{bmatrix}\left\{\begin{matrix}\mathbf{P}\\ \mathbf{Q}\\ \mathbf{0}\end{matrix}\right\} = \left\{\begin{matrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\end{matrix}\right\}, \tag{3}$$

where $\mathbf{K}$ is the stiffness matrix and $\mathbf{C} = \mathbf{K}^{-1}$ is the compliance matrix. Equations (1:(i), 3) then imply that $\mathbf{C}_{nt}\mathbf{Q} = -\mathbf{C}_{nn}\mathbf{P}$ and since $\mathbf{C}_{nn}$ is symmetric and positive definite (Γ_D is not null), we obtain $-(\mathbf{C}_{nn})^{-1}\mathbf{C}_{nt}\mathbf{Q} = \mathbf{P}$. Note that this equation depends on the entire finite element discretization, not only on the discretization near the contact zone, since it involves the coefficients of $\mathbf{K}^{-1}$, the inverse of the stiffness matrix. Substituting in (1:(ii)), we then obtain

$$-(\mathbf{C}_{nn})^{-1}\mathbf{C}_{nt}\mathbf{P} = \frac{1}{f}\mathbf{P}\ . \tag{4}$$

In other words, slip solutions of the assumed form exist if and only if $1/f$ is an eigenvalue of equation (4).

3.1 Relation to the wedging problem

If the configuration of Fig.2 can exist in a wedged state, the corresponding values of $\mathbf{u}$, $\mathbf{P}$, $\mathbf{Q}$ must satisfy the conditions

$$|Q_i| \leq f P_i \; ; \quad u_i \geq 0 \; ; \quad P_i \geq 0 \; ; \quad u_i P_i = 0 \; , \tag{5}$$

where $\mathbf{u}$, $\mathbf{P}$, $\mathbf{Q}$ are related through equations (1:(i), 2, 3).

Now suppose that the eigenvalue problem (4) has at least one real eigenvalue $1/f_c$ and that the corresponding eigenfunction satisfies the condition that the P_i $(i = 1, N)$ all have the same sign, so that we can choose the sign of the arbitrary multiplier on the eigenfunction so as to satisfy the condition $P_i \geq 0$. It follows that the same set of displacements and contact tractions will satisfy the conditions (5) for wedging if $f \geq f_c$.

Equation (4) is an $N \times N$ linear eigenvalue equation and it will always have N eigenvalues and associated eigenfunctions. However, the matrix $(\mathbf{C}_{nn})^{-1}\mathbf{C}_{nt}$ is not generally symmetric and hence there is no guarantee that all or indeed any of the eigenvalues will be real. Since the eigenvalues represent the coefficient of friction, complex values have no physical significance. Furthermore, even if real eigenvalues exist, there is no guarantee that the associated eigenfunctions satisfy the condition that the P_i $(i = 1, N)$ all have the same sign. It is fairly easy to generate examples in which these restrictions prevent the eigenvalue problem from defining a physically meaningful value of f_c. A simple example is the square block of Fig.2, discretized such that there exist only two nodes in Γ_C located at the two lower corners A, B [15].

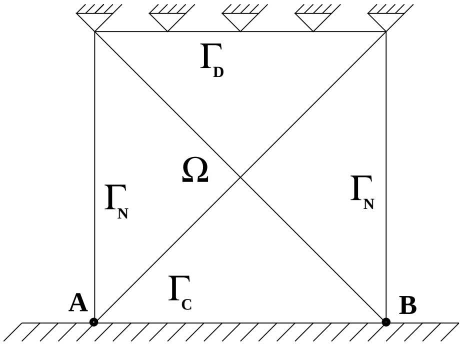

Fig. 2. Rectangular geometry

3.2 A generalized non-linear eigenvalue problem

These restrictions do not necessarily imply that wedged states are impossible when the corresponding eigenvalue problem fails to yield a physically acceptable eigenfunction. For example, a simple investigation of the example of Fig.2

shows that physically acceptable wedged states can occur (a) if both contact nodes are displaced in the same direction (say to the right) and the right hand node is allowed to separate, or (b) if both nodes are displaced towards the centre of the contact region and remain in contact [15].

These considerations suggest the definition of a generalized non-linear eigenvalue problem through the conditions

$$|Q_i| = f_c P_i \; ; \quad u_i \geq 0 \; ; \quad P_i \geq 0 \; ; \quad u_i P_i = 0 \; . \tag{6}$$

Comparison with (5) shows that the existence of a real eigenvalue f_c to this problem is a sufficient condition for wedged states to exist for $f \geq f_c$ and hence the minimum real eigenvalue $f_c^{\min}$ represents at least an upper bound for the critical coefficient of friction required for wedging to occur.

It is less clear whether this condition is also necessary — i.e. whether the existence of a wedged state with a given value of f implies the existence of a real eigenvalue for the problem (6) with $f_c \leq f$. However, the following thought experiment suggests that this is in fact the case. Suppose we start with the given wedged state and imagine some mechanism such as the gradual infusion of lubricant whereby the coefficient of friction can be gradually reduced. Eventually a value of f will be reached at which one node will start to slip and further reduction in f will transfer load to other nodes causing them to slip.

To place this argument on a more rigorous footing, consider an intermediate state in which a subset Γ_S of Γ_C is in a state of slip with coefficient of friction f, whilst the remainder $\Gamma_C - \Gamma_S$ remains stuck. In other words, we suppose the existence of a non-trivial solution to the problem

$$Q_i = \pm f P_i \quad \in \Gamma_S \tag{7}$$

$$|Q_i| < f P_i \quad \in \Gamma_C - \Gamma_S \tag{8}$$

$$u_i \geq 0 \; ; \quad P_i \geq 0 \; ; \quad u_i P_i = 0 \quad \in \Gamma_C \; . \tag{9}$$

where the sign taken in (7) depends on the direction of slip and may be different at different nodes.

Suppose we now reduce the coefficient of friction by an infinitesimal increment Δf. If the conditions after this increment are to satisfy (7, 9) with f replaced by $f - \Delta f$ and no change in state at any node, the corresponding incremental changes in P, Q, u, v must satisfy the conditions

$$Q_i - \Delta Q_i = \pm(f - \Delta f)(P_i - \Delta P_i) \quad \in \Gamma_S \tag{10}$$

$$\Delta v = 0 \quad \in \Gamma_C - \Gamma_S \tag{11}$$

$$\Delta u = 0 \quad \in \Gamma_C \tag{12}$$

and since Δf is infinitesimal, we can drop second order small quantities in (10), giving

$$\Delta Q_i \mp f \Delta P_i = \pm \Delta f P_i \quad \in \Gamma_S \; . \tag{13}$$

Equations (11–13) define a linear problem for the incremental fields, since P_i is assumed known. We can therefore solve this problem and add the resulting increment in P, Q, u, v to the original state to obtain the new state, which will not violate the inequalities in (9), provided Δf is sufficiently small.

By making a succession of small but finite reductions Δf, we can monitor each inequality and determine which is the first to be violated as f is reduced. When a violation is detected, we can also solve for the exact value of Δf at which this node(s) just reaches the limiting condition ($P_i = 0$ for a 'slip' node or $|Q_i| = fP_i$ for a 'stick' node). We can then change the assumption at the node(s) in question and proceed.

Using this algorithm, as long as any nodes remain stuck, equations (7–9) define a wedged state. The limiting condition arises when the last node(s) in $\Gamma_C - \Gamma_S$ just reaches the slip condition. At this point, all nodes are either separated or slipping and hence the final wedged state is a solution of the generalized eigenvalue problem (6). This procedure could be applied for any initial wedged configuration and hence we conclude that the smallest eigenvalue f_c of (6) is also the critical coefficient of friction above which wedged states can occur.

3.3 Solution of the generalized eigenvalue problem

We now turn our attention to possible strategies for solving the non-linear eigenvalue problem (6). We first note that each contact node must be in one of three states (forward slip, backward slip or separation) and hence the number of possible combinations of states with N contact nodes is 3^N. Each of these combinations defines a classical linear eigenvalue problem, so we might solve each in turn and search the resulting solutions for the minimum real eigenvalue whose eigenfunction satisfies the inequality constraints with an appropriate choice for the sign of the multiplying constant.

This method proceeds by exhaustion and will generate the critical coefficient of friction for wedging from a finite number of numerical calculations. However, it is likely to be prohibitively computer intensive for large systems. An alternative approach is to develop an iterative method using strategies drawn from the corresponding unilateral contact problem. For example, suppose we solve the linear eigenvalue problem assuming unidirectional slip at all nodes and obtain a real eigenvalue associated with an eigenfunction that fails the requirement that the normal tractions be all of the same sign. We might then reformulate the linear eigenvalue problem making the assumption of separation at the nodes where the tractions are tensile. There is of course no guarantee that such an iterative procedure would be convergent, but any solutions so obtained could at least be used to define an upper bound for the critical coefficient of friction.

3.4 Numerical experiments

Yet another option, which we explore here numerically, is to program the 'relaxation' algorithm of section 3.2. We first determine an initial wedged condition using prescribed values of $\mathbf{v}$ and a suitably high coefficient of friction and then try to reduce the friction coefficient using the following procedure:-

(i) Identify the node j at which the maximum ratio between tangential and normal traction is attained.
(ii) Relax the tangential displacement v_j at this node and solve the elasticity problem with no loads, a prescribed v and unilateral contact conditions (6) until another node reaches the same limiting condition.
(iii) Continue relaxation until all the nodes are at the same limiting condition, corresponding to slip and identify the limiting value of f.

We first consider the triangle Ω of vertices $A = (0,0)$, $B = (1,0)$ and $C = (0.6, 0.3)$ and we set $\Gamma_D =]B, C[$, $\Gamma_N =]A, C[$, $\Gamma_C =]A, B[$. The body Ω lies on a rigid foundation, the half-space delimited by the straight line (A, B). Moreover we assume that $\nu = 0.2$. We consider a mesh comprising 10 contact elements and 10 contact nodes (the node B is supposed to belong to Γ_D in the computations). We choose an initial 'constant' tangential displacement $v = (0.05, ..., 0.05)$ (the 10 nodes move to the right with a displacement of 0.05) and we solve the elasticity problem with no loads and unilateral contact conditions (6). The range of the ratio between tangential and normal traction for this configuration is [0.715,4.021], so it corresponds to a wedged state for $f \geq 4.021$. We apply the above procedure with a relaxation coefficient of 0.999. After 1000 iterations the range of the ratio is [0.715,3.146] and after 10000 iterations it is [1.991,2.006], showing that wedging occurs if $f \geq 2.006$. We observe that the limiting value is $f = 2$ which precisely solves the associated linear eigenvalue problem corresponding to slip in the continuous case [17]. Moreover the limiting field v is no longer constant on the 10 nodes (and vanishing at B) but linear and the stress field is constant in Ω.

Next we consider the geometry of a tapered joint (with $\nu = 0.1$) which is a geometry conducive to wedging. Using symmetry conditions, the tapered joint can be divided into two parts and the geometry to be considered is a trapezium with two right angles. As before we choose a constant initial displacement $v = (0.1, ..., 0.1)$, for which the range of the tangential/normal traction ratio is found to be [-0.0418,0.2128]. We used a relaxation coefficient of 0.99999. After 10000 iterations we obtain a wedged configuration for $f \geq 0.1002$ and the range of the ratio is [0.0995,0.1002]. It seems that the limiting value is $f = 0.1$ which is precisely equal to $\tan(\theta)$ (2θ is the angle of the entire joint).

The Von Mises stress fields after 1 and 10000 iterations are depicted in Fig. 3. The limiting stress field increases from the left to the right and admits physically relevant isolines which are parallel to the sides of the trapezium.

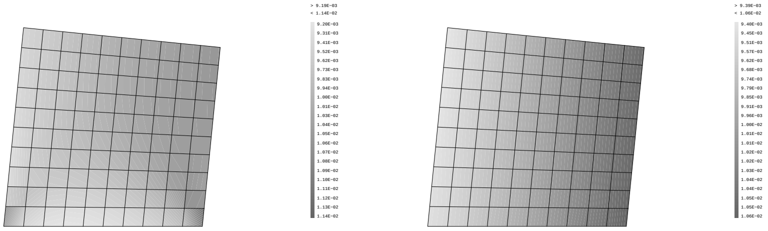

Fig. 3. Initial and limiting Von-Mises stress fields

4 Conclusions

This paper represents a first attempt to quantify the coefficient of friction above which wedging is possible for a two-dimensional elastic contact problem. We have demonstrated that this is also the solution of a certain non-linear eigenvalue problem. Future work will be directed at testing various suggestions for the solution of this problem.

References

1. Nečas J, Jarušek J, Haslinger J (1980) Boll Unione Mat Ital 17:796–811.
2. Jarušek J (1983) Czechoslovak Math J 33:237–261.
3. Kato Y (1987) Japan J Appl Math 4:237–268.
4. Eck C, Jarušek J (1998) Math Models Methods Appl Sci 8:445–468.
5. Andersson L-E (2000) Appl Math Optim 42:169–202.
6. Rocca R, Cocu M (2001) Int J Engrg Sci 39:1233–1255.
7. Oden J, Pires E (1983) ASME J Appl Mech 50:67–76.
8. Haslinger J (1983) Math Meth Appl Sci 5:422–437.
9. Kikuchi N, Oden JT (1988) Contact problems in elasticity: A study of variational inequalities and finite element methods, SIAM, Philadelphia.
10. Mosemann H, Wahl FM (2001) IEEE Trans Robotics Automation 17:709–718.
11. Sturges RH, Laowattana S (1996) J Mech Des 118:99–105.
12. Klarbring A (1984) Contact problems with friction, Doctoral Dissertation, Linköping University Linköping Sweden
13. Klarbring A (1990) Ing Arch 60:529–541
14. Cho H, Barber JR (1998) Math Comp Modeling 8:37–53
15. Barber JR, Hild P (2004) Proc. ASME/STLE Int. Joint Tribology Conference, Long Beach, CA.
16. Hassani R, Hild P, Ionescu I, Sakki N-D (2003) Comput Methods Appl Mech Engrg 192:4517–4531.
17. Hild P (2004) Q J Mech Appl Math 57:235–245