3. One way of satisfying the compatibility equations in the absence of rotation is to define the components of displacement in terms of a potential function ψ through the relations

$$u_x = \frac{\partial\psi}{\partial x} \;\; ; \;\; u_y = \frac{\partial\psi}{\partial y} \;\; ; \;\; u_z = \frac{\partial\psi}{\partial z} \; .$$

Use the stress-strain relations to derive expressions for the stress components in terms of ψ.

Hence show that the stresses will satisfy the equilibrium equations in the absence of body forces if and only if

$$\nabla^2\psi = \text{constant} \; .$$

4. Plastic deformation during a manufacturing process generates a state of residual stress in the large body $z>0$. If the residual stresses are functions of z only and the surface $z=0$ is not loaded, show that the stress components $\sigma_{yz}, \sigma_{zx}, \sigma_{zz}$ must be zero everywhere.

5. By considering the equilibrium of a small element of material similar to that shown in Figure 1.2, derive the three equations of equilibrium in cylindrical polar coördinates r, θ, z.

6. In cylindrical polar coördinates, the strain-displacement relations for the 'in-plane' strains are

$$e_{rr} = \frac{\partial u_r}{\partial r} \;\; ; \;\; e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \;\; ; \;\; e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \; .$$

Use these relations to obtain a compatibility equation that must be satisfied by the three strains.

7. If no stresses occur in a body, an increase in temperature T causes unrestrained thermal expansion defined by the strains

$$e_{xx} = e_{yy} = e_{zz} = \alpha T \;\; ; \;\; e_{xy} = e_{yz} = e_{zx} = 0 \; .$$

Show that this is possible only if T is a linear function of x, y, z and that otherwise stresses must be induced in the body, regardless of the boundary conditions.

8. If there are no body forces, show that the equations of equilibrium and compatibility imply that

$$(1+\nu)\frac{\partial^2\sigma_{ij}}{\partial x_k\partial x_k} + \frac{\partial^2\sigma_{kk}}{\partial x_i\partial x_j} = 0 \; .$$