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The application of asymptotic solutions to contact problems characterised by logarithmic singularities

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Abstract

We give the contact pressure distribution near a contacting wedge having a slightly rounded form adjacent to a discontinuity in surface profile. It is shown that, well away from the rounding the pressure is logarithmic in form, just as it is near the apex of a sharp wedge. This pair of solutions may then be used to 'patch in' a roundness correction relevant to *any* punch having a discontinuous gradient. Further, it is noted that the multiplier on the logarithm term is pre-determined by the change in gradient. This process is applied to a finite, slightly blunt wedge, where the exact answer is known, and to a wheel having a worn flat. The agreement with the exact solution in the former case is seen to be very good.
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1. Introduction

This paper is intended to provide a tool for determining the local contact pressure at a point within a contact where there is an internal (remote from the edges) discontinuity in the slope of the surface of the indenting component, but with a degree of local rounding. For example, Fig. 1(a) depicts a wheel having a 'flat' so that there is a local discontinuity of slope at the end of the worn flat (point A) and, as is well known, this will cause a local (weak) logarithmic singularity in the contact pressure (Sackfield et al., 2006, 2007).

For *all* contact geometries containing a discontinuity in surface slope away from the edges, the pressure distribution in the neighbourhood of the apex is in the form (Truman et al., 1995; Sackfield et al., 2005):

$$p(x) = -\frac{2\phi}{\pi A}\ln|x| + \cdots, \tag{1}$$

where ϕ is the change in slope at this point, x is measured from the discontinuity, and A is the composite compliance of the bodies ($= (1-\nu)/\mu$ if the bodies are elastically similar, and $= 1/2\mu$ if the indenter is rigid and the indented

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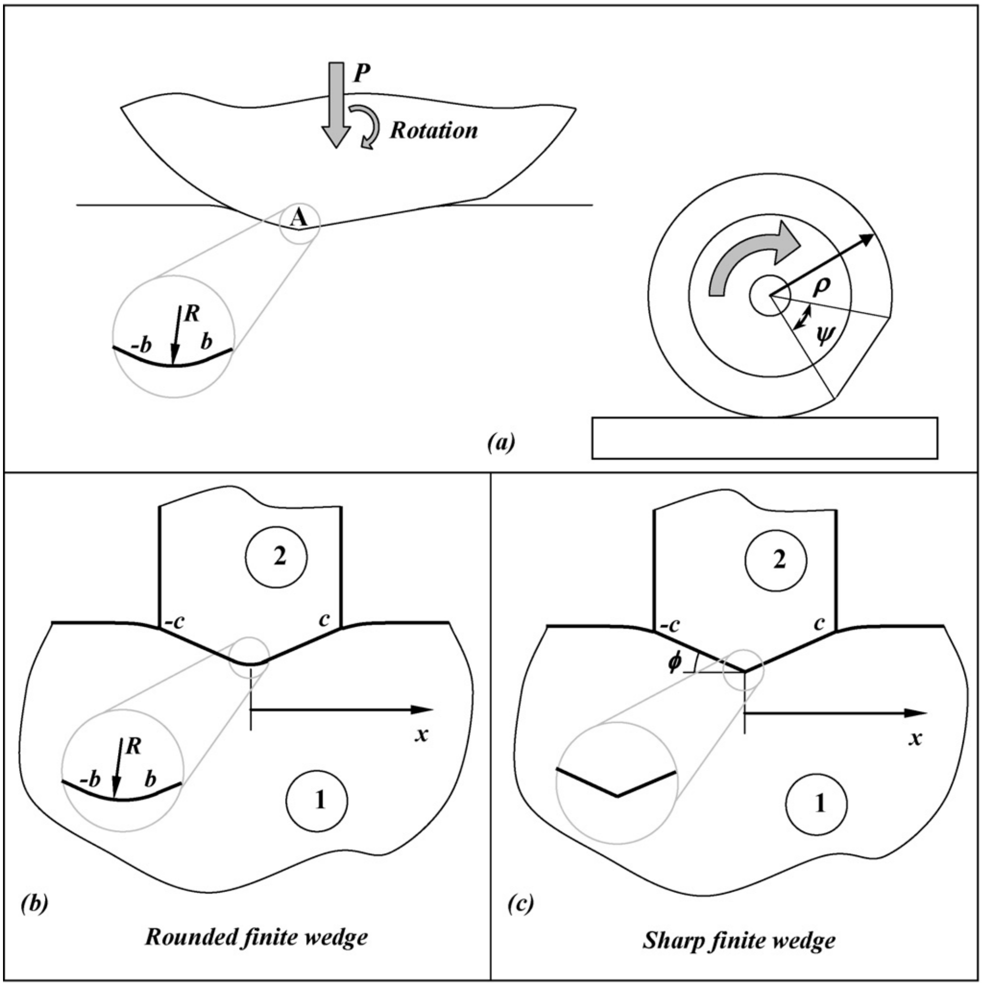

Fig. 1. Example problem (a) and schematic of the (b) rounded and (c) sharp finite wedge problem to be used to verify the validity of the newly developed asymptotic formulation.

body incompressible). Here μ is the shear modulus and ν the Poisson's ratio. This shows that the magnitude of the state of stress, i.e. the multiplier on the logarithm, is a constant which depends only on the elastic constants of the contacting pair, and the change of slope of the contact face. It is independent of the applied load on the punch, whatever its shape, and regardless of whether the contact as a whole is complete or incomplete in nature.

In reality, no atomically sharp discontinuity will exist, and the effects of plasticity and wear will rapidly 'round off' the sharp corner. It is the intention, here, to develop a procedure for taking the 'sharp' form of the profile, which will give a well-defined logarithmic behaviour, and to provide a means of 'patching in' a solution for the effect of the rounding. This will be of practical use in several problems; those, like the wheel with a flat, which are just about viable as a closed-form analytical problem when the rounding element itself is omitted, and those which are so complex that a numerical solution is, in case, required. However, in order that the procedure to be developed can be applied to a problem in which the exact answer is known, we shall first consider the problem of indentation by a rigid punch whose profile is shown in Fig. 1(b)[1], i.e. a punch of finite width whose front face is a wedge having a small external angle, ϕ, but with a tip radius R extending from $-b$ to b, and where the contact force is sufficiently large for complete contact to be maintained. This will be done by considering the solution for the perfectly 'sharp' version of the same problem, Fig. 1(c), and applying the modification to the surface contact pressure, to be derived.

[1] Note that the equivalent semi-infinite problem has been solved in closed form by Ciavarella et al. (1998).

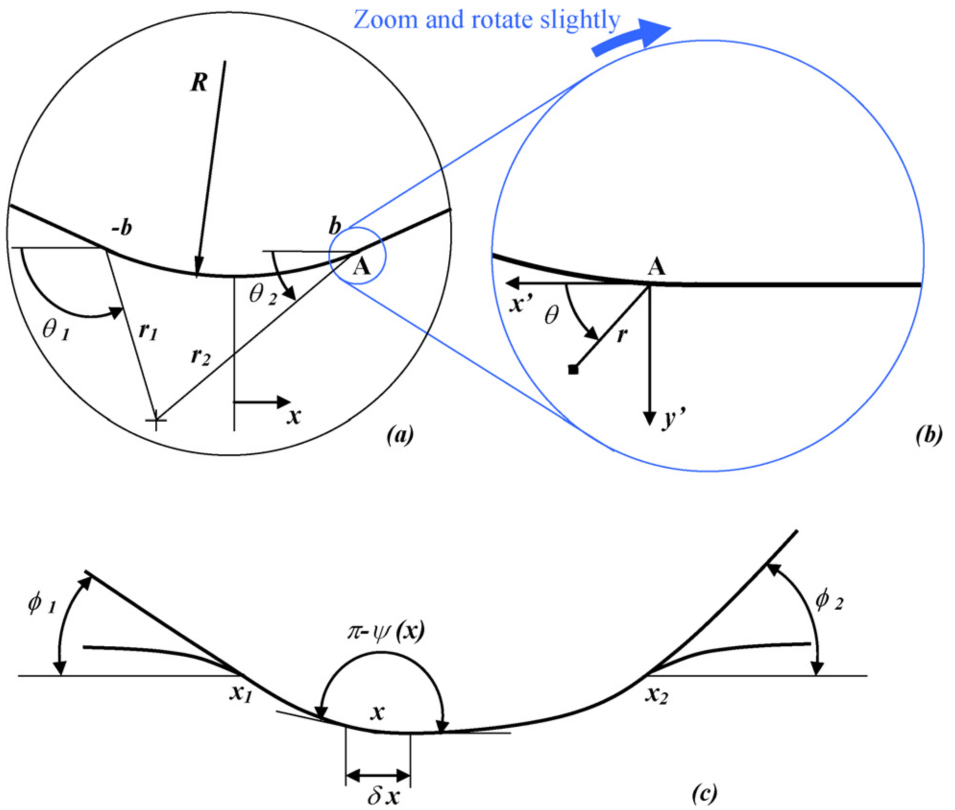

Fig. 2. (a) Schematic of the asymptotic problem to be solved (see also inset to Fig. 1(b)) and (b) asymptote to the asymptote. It should be noted that these schematics are not to scale as the curvature of the rounded portions of the wedge should be very large for half-plane theory to hold. (c) Schematic of a generic smooth rounded profile and its representation by a series of infinitesimal discontinuities for the solution of the problem by superposition using the sharp corner solution as a Green's function.

2. The rounded apex

The problem we wish to address is to find the contact pressure, $p^*(x)$, and attendant state of stress near the internal rounded apex, Fig. 2(a). We already know that, when the observation point is well away from the rounding the contact pressure should tend asymptotically to the form given in Eq. (1). We may usefully consider an even simpler problem, viz. frictionless contact between the infinite rigid indenter and half-plane shown in Fig. 2(b), where we shall place the origin of a polar coordinate (r, θ) set, used to define deformation on the surface of the half-plane, at the transition point between the straight, inclined face and the radiused part of the face of the indenter. The fundamental asymptotic problem is defined by the boundary conditions

$$\frac{\partial u_\theta}{\partial r} = 0; \qquad \sigma_{r\theta} = 0 \quad \text{on } \theta = -\pi \tag{2}$$

and

$$\frac{\partial u_\theta}{\partial r} = \frac{r}{R}; \qquad \sigma_{r\theta} = 0 \quad \text{on } \theta = 0. \tag{3}$$

An Airy stress function, usually associated with Williams' wedge problem, here specialised to a half-plane, and sufficiently general to solve the above problem is

$$\Phi = r^{\lambda+1}\{A_1 \cos(\lambda+1)\theta + A_2 \cos(\lambda-1)\theta + A_3 \sin(\lambda+1)\theta + A_4 \sin(\lambda-1)\theta\}. \tag{4}$$

The relevant corresponding stress and displacement components are

$$\sigma_{r\theta} = r^{\lambda-1}\{A_1\lambda(\lambda+1)\sin(\lambda+1)\theta + A_2\lambda(\lambda-1)\sin(\lambda-1)\theta$$

$$- A_3\lambda(\lambda+1)\cos(\lambda+1)\theta - A_4\lambda(\lambda-1)\cos(\lambda-1)\theta\big\}, \tag{5}$$

$$2\mu u_\theta = r^\lambda\big\{A_1(\lambda+1)\sin(\lambda+1)\theta + A_2(\kappa+\lambda)\sin(\lambda-1)\theta - A_3(\lambda+1)\cos(\lambda+1)\theta - A_4(\kappa+\lambda)\cos(\lambda-1)\theta\big\} \tag{6}$$

so that the only inhomogeneous boundary condition implies the use of $\lambda = 2$ and we then have

$$\sigma_{r\theta} = r\big\{6A_1\sin(3\theta) + 2A_2\sin\theta - 6A_3\cos(3\theta) - 2A_4\cos\theta\big\}, \tag{7}$$

$$2\mu u_\theta = r^2\big\{3A_1\sin(3\theta) + A_2(\kappa+2)\sin\theta - 3A_3\cos(3\theta) - A_4(\kappa+2)\cos\theta\big\}, \tag{8}$$

where κ is the Kolosov's constant ($\kappa = 3 - 4\nu$ in plane strain). The boundary conditions (9), (11) yield the algebraic equations

$$3A_3 + A_4(\kappa+2) = 0, \tag{9}$$

$$6A_3 + 2A_4 = 0, \tag{10}$$

$$3A_3 + A_4(\kappa+2) = C, \tag{11}$$

$$6A_3 + 2A_4 = 0, \tag{12}$$

but clearly there is no solution of these equations, since (9), (11) are incompatible.

The solution to this difficulty is discussed in Section 10.3 of Barber (2002). We need to add one or more special solutions, obtained by differentiation of (4) with respect to λ, before setting $\lambda = 2$. In fact, this was to be anticipated, as we knew that the traction distribution must contain a logarithm, and this is precisely the result of the differentiation with respect to λ. We therefore choose

$$\Phi = r^3\big[a_1\cos\theta + a_2\cos 3\theta + a_3\sin\theta + a_4\sin 3\theta + a_5(\ln r\cos\theta - \theta\sin\theta) + a_6(\ln r\sin\theta + \theta\cos\theta) + a_7(\ln r\cos 3\theta - \theta\sin 3\theta) + a_8(\ln r\sin 3\theta + \theta\cos 3\theta)\big] \tag{13}$$

and it is shown in Appendix A that the corresponding contact pressure must be of the form

$$p_{\text{asy}}(x') = -\frac{x'}{\pi RA}\big[\ln|x'| - const\big]. \tag{14}$$

This is the final solution for a single straight-rounded transition in the profile of the punch given in Fig. 2(b).

We can now use superposition to determine the contact pressure for the original finite rounding problem, Fig. 2(a), with the result

$$p^*(x) = f(x+b) - f(x-b), \tag{15}$$

where

$$f(x) = -\frac{\phi}{\pi Ab}\big[x\ln|x| - x\big]. \tag{16}$$

If follows that

$$p^*(x) = -\frac{2\phi}{\pi A}\left[\frac{1}{2}\left(1-\frac{x}{b}\right)\ln|b-x| + \frac{1}{2}\left(1+\frac{x}{b}\right)\ln|b+x| - 1\right], \tag{17}$$

where we have chosen in Eq. (14) the constant to be unity ($const = 1$) so that when $x \to \infty$, Eq. (17) becomes:

$$p^*(x) \to -\frac{2\phi}{\pi A}\ln|x|, \tag{18}$$

i.e. the outer form of this solution is precise the same as the inner form of the characteristic pressure distribution for the perfectly sharp solution, permitting very straightforward matching of the two. In order to apply this result we merely replace the term $\frac{2\phi}{\pi A}\ln|x|$ in the 'sharp' form of any finite problem by $p^*(x)$, or, equivalently, we can add the *universal* corrective term

$$p^c(x) = p^*(x) + \frac{2\phi}{\pi A}\ln|x| = -\frac{2\phi}{\pi A}\left[\frac{1}{2}\left(1-\frac{x}{b}\right)\ln\left|\frac{b}{x}-1\right| + \frac{1}{2}\left(1+\frac{x}{b}\right)\ln\left|\frac{b}{x}+1\right| - 1\right], \tag{19}$$

which may be applied to *any* contact problem involving an internal slope discontinuity, in a very straightforward manner.

The state of stress in the neighbourhood of the rounded apex shown in Fig. 2(a) may be found by superposition of the Airy stress function (A.6) for the reduced problem of Fig. 2(b) given in Appendix A, with the result

$$\Phi^c(x, y) = \Phi_{\mathrm{asy}}(r_1, \theta_1) - \Phi_{\mathrm{asy}}(r_2, \theta_2), \tag{20}$$

where

$$r_1^2 = (x+b)^2 + y^2, \qquad r_2^2 = (x-b)^2 + y^2, \tag{21}$$

$$\tan(\theta_1) = \frac{y}{x+b}, \qquad \tan(\theta_2) = \frac{y}{x-b}. \tag{22}$$

It follows that the corresponding complete elastic field for the rounded contact problem of interest can be recovered from that for the corresponding sharp problem by adding the Airy function (20).

An alternative derivation of the contact pressure distribution, achievable again from first principles, is to use the sharp wedge solution (1) as a Green function. This has the advantage that it may be used to represent local rounding with curvature varying in an arbitrary way. The formulation, described in Appendix B, allows corroboration of the asymptotic solution given above.

3. Application of corrective solution

The asymptotic matching procedure in this problem is much simpler than in the case of power-law singularities, since in the latter case it is necessary to extract a generalised stress intensity factor (Dini and Hills, 2003, 2004; Hills et al., 2004). In the present case, the multiplier on the singular term is already known since it is defined by the angle of the corner and the material properties ($2\phi/\pi A$). Thus, all we need to do is to write down the expression for the sharp punch pressure distribution, after which the rounded solution is obtained by merely superposing the universal corrective term (19).

For the sharp, finite punch problem, depicted in Fig. 1(c), the contact pressure, $p(x)$, along the interface of a finite rigid punch of half-width, c, and with a discontinuity in slope at $x = 0$ of 2ϕ is (see Appendix C)

$$p_{\mathrm{sharp}}(x) = -\frac{2\phi}{\pi A}\left(\ln\left|\frac{1-\sqrt{1-x/c}/\sqrt{1+x/c}}{1+\sqrt{1-x/c}/\sqrt{1+x/c}}\right| + \frac{1+Ap_0/\phi}{\sqrt{1-(x/c)^2}}\right), \quad \frac{-1<x}{c<+1}, \tag{23}$$

where $p_0 = P/2c$. The solution to the problem is valid providing that the load is sufficiently high for contact to be maintained over the entire front face, and this requires that

$$P > \frac{2c\phi}{A}. \tag{24}$$

The solution to the slightly rounded problem is therefore found simply by adding the corrective term (Eqs. (19) to (23)). In this instance, it is worth looking at the resulting expression in a little detail, because it apparently still contains a logarithmic singularity at the origin. However, we note that the logarithmic term in the expression for the finite, sharp-nosed punch may be written in the form

$$\ln\left|\frac{1-\sqrt{1-x/c}/\sqrt{1+x/c}}{1+\sqrt{1-x/c}/\sqrt{1+x/c}}\right| = \ln|x| - \ln\left|\frac{c}{2}\right| - 2\ln\left(\sqrt{1+x/c}+\sqrt{1-x/c}\right), \tag{25}$$

and this gives a final form for the approximate solution of

$$\begin{aligned} p_{\mathrm{corrected}}(x) = &-\frac{2\phi}{\pi A}\left[\frac{1+Ap_0/\phi}{\sqrt{1-(x/c)^2}} - 2\ln\left(\sqrt{1+x/c}+\sqrt{1-x/c}\right)\right] \\ &-\frac{2\phi}{\pi A}\left[\frac{1}{2}\left(1-\frac{x}{b}\right)\ln\left|\frac{2(b-x)}{c}\right| + \frac{1}{2}\left(1+\frac{x}{b}\right)\ln\left|\frac{2(b+x)}{c}\right| - 1\right]. \end{aligned} \tag{26}$$

Normally, this algebraic step would be omitted, and the corrective term simply added-in, but it is included here to illustrated the cancelling of the logarithmic singularity at the origin. For comparison, the exact answer for the rounded punch, derived in the conventional manner in Appendix D, is given by

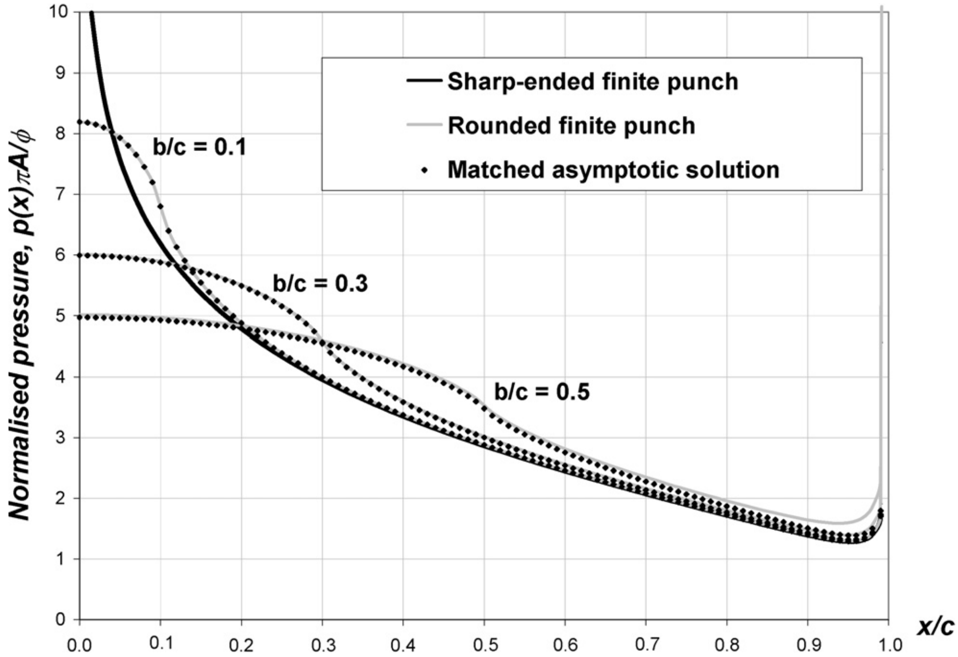

Fig. 3. Comparison between the contact pressure distribution for the problem shown in Fig. 1(c) as obtained from the exact solution, and adding the corrective term (19) for different values b/c. The dimensionless load is $Ap_0/\phi = -1.1$.

$$p_{\text{rounded}}(x) = -\frac{2\phi}{\pi A}\left\{\left(1-\frac{x}{b}\right)\frac{F(c,b,x)}{2} + \left(1+\frac{x}{b}\right)\frac{F(c,-b,x)}{2} + \frac{1}{2\sqrt{1-(x/c)^2}}\left[\sqrt{1-\left(\frac{b}{c}\right)^2} + \frac{2(x/c)^2-1}{b/c}\sin^{-1}\left(\frac{b}{c}\right)\right] + \frac{\frac{Ap_0}{\phi}}{\sqrt{1-(x/c)^2}}\right\}. \tag{27}$$

Note that inequality (24) needs to be satisfied for this problem also, in order for completeness to be achieved.

3.1. Results

Fig. 3 displays the results for the radiused finite punch shown in Fig. 1(b), together with the solution for the sharp form of the punch shown in Fig. 1(c) for an example applied load. Also given in the same figure are the results found by following the process described, and which, in this instance, explicitly leads to Eq. (26). The difference between the exact answer and the solution found by applying a correction to the solution to the sharp form of the indenter is very small indeed, even when the radius extends as far as a half of the half-width of the punch. In fact, on this scale, it is hard to detect the difference between the exact solution and the one obtained by applying the corrective term, over the whole extent of the punch, except the extreme edge.

4. Further aspects

Problems such as the wheel with a flat, mentioned at the outset, Fig. 1(a), include the additional complication that, in the neighbourhood of the discontinuity in slope, the indentation is not symmetrical. However, this need introduce no further complications to the process of correcting for the presence of a small radius. To illustrate the procedure, consider the tilted sharp-nosed punch shown in Fig. 4(a). It has been demonstrated in Churchman et al. (2006) that the contact pressure distribution for this problem is given by

$$p_{\text{tilted}}(x) = -\frac{2\phi}{\pi A}\ln\left|\frac{1-\sqrt{1-x/d_1}/\sqrt{1+x/d_2}}{1+\sqrt{1-x/d_1}/\sqrt{1+x/d_2}}\right|, \quad -d_1 < x < d_2, \tag{28}$$

and the corrective term (Eq. (19)) may simply be added, without further modification. Fig. 5 shows the contact pressure distribution for the tilted, sharp punch shown in Fig. 4(a), given by Eq. (28) and the solution to the rounded equivalent

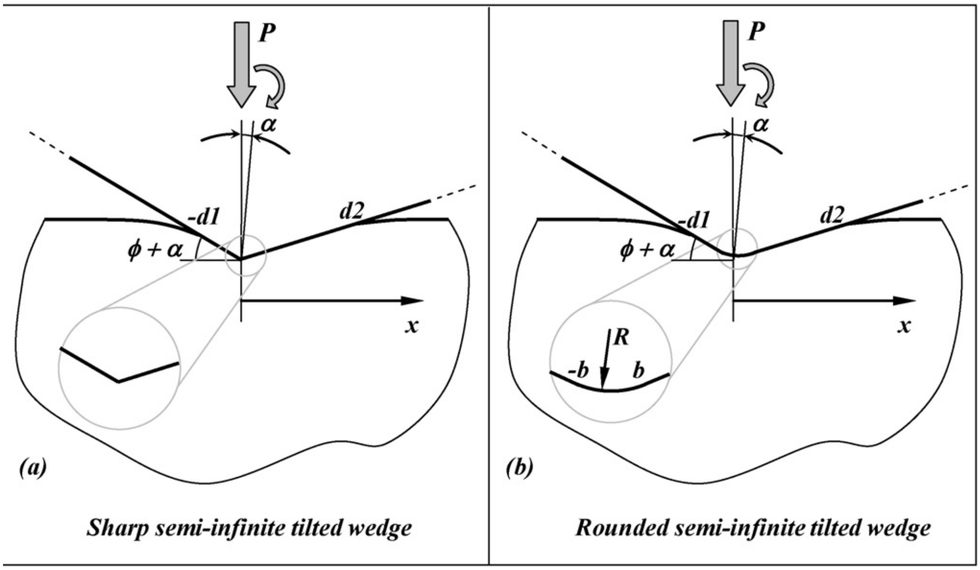

Fig. 4. Schematic of the tilted wedge problem to which the asymptotic matching has also been applied using the rounded logarithmic solution.

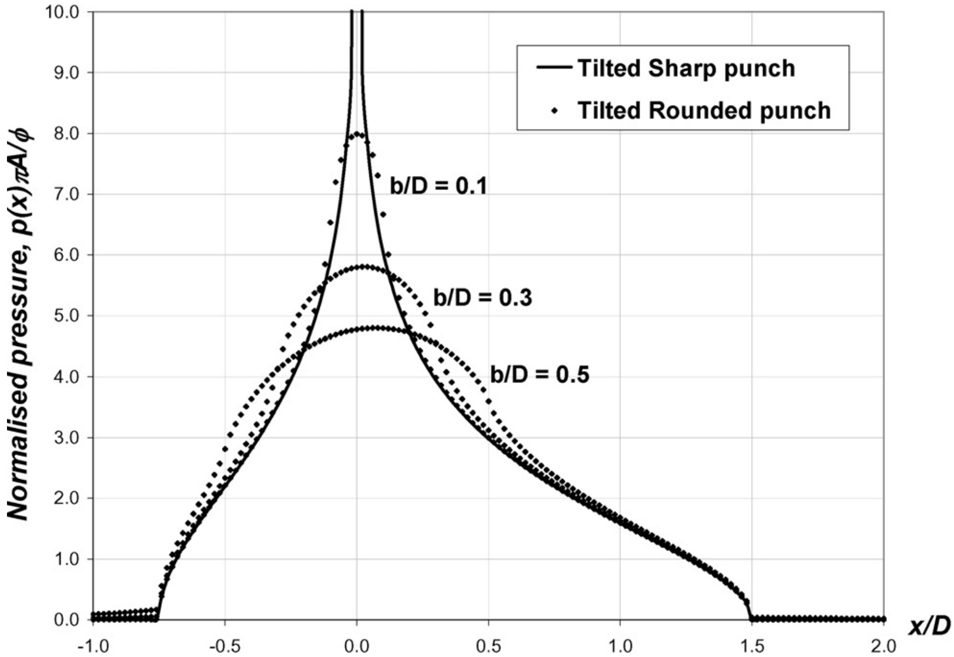

Fig. 5. Comparison between the contract pressure distribution for the tilted, sharp punch shown in Fig. 4(a), and the pressure distributions for the matched rounded solution for various dimensionless rounding parameter b/D and $d_2/d_1 = 2$.

problem, Fig. 4(b), achieved through asymptotic matching for the for various angles of tilt (or equivalently ratios d_2/d_1) and rounding dimensionless parameter b/D, with

$$D = \frac{2d_1d_2}{d_1 + d_2}. \tag{29}$$

A further aspect of this problem concerns the prediction of the peak contact pressure in the presence of rounding, which is given by

$$p_{\max} = \left(p_{\text{sharp}}(x) + \frac{2\phi}{\pi A} \ln|x| \right)\Big|_{x=0} - \frac{2\phi}{\pi A}\big[\ln(b) - 1\big], \tag{30}$$

where $p_{\text{sharp}}(x)$ is the solution for the corresponding 'sharp' form of the contact, from which the logarithmic term is then subtracted, and the correction subsequently applied. This gives an 'exact' value (within the spirit of the corrective solution) whenever the discontinuity in slope is symmetrical, and applies approximately when this condition is not satisfied.

5. Conclusion

The solution for the difference in contact pressure distributions between a sharp discontinuity and a slightly rounded discontinuity in the slope at an interior point with a general plane contact has been found, employing half-plane theory. This has the effect of attenuating a weak logarithmic singularity to produce a finite local peak, whose magnitude can easily be quantified. The corrective solution can equally be applied to problems where the discontinuity in slope is symmetrical in x, and those where it is not. Although, strictly speaking, the solution can only be expected to work when; (a) the discontinuity in slope, ϕ, is small, and (b) the straight lengths adjacent to the tip radius are long compared with the radius itself, in the example problem chosen only the former is a practical limitation, and that results from using half-plane theory.

Appendix A. Corrective solution

An Airy stress function sufficiently general to solve the problem is

$$\begin{aligned}\Phi = r^3\big[&a_1\cos\theta + a_2\cos 3\theta + a_3\sin\theta + a_4\sin 3\theta + a_5(\ln r\cos\theta - \theta\sin\theta) \\ &+ a_6(\ln r\sin\theta + \theta\cos\theta) + a_7(\ln r\cos 3\theta - \theta\sin 3\theta) + a_8(\ln r\sin 3\theta + \theta\cos 3\theta)\big].\end{aligned} \tag{A.1}$$

Now

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right\} \tag{A.2}$$

and Eqs. (2) and (3) (see main text) provide four boundary conditions leading to three independent equations for the coefficients.

Similarly the displacement field corresponding to the stress function (A.1) may be found from Table 9.1 (Barber, 2002)

$$\begin{aligned}2\mu u_\theta = r^2\big\{&-a_3(\kappa+2)\cos\theta - 3a_4\cos 3\theta + a_5(\kappa+2)\theta\cos\theta - a_6\big[\cos\theta + (\kappa+2)\ln r\cos\theta\big] \\ &+ 3a_7\theta\cos 3\theta - a_8[\cos 3\theta + 3\ln r\cos 3\theta]\big\}\end{aligned} \tag{A.3}$$

and imposing the displacement boundary conditions at $\theta = 0, -\pi$ gives us three independent further equations for the coefficients.

We therefore solve a system of six simultaneous equations for the coefficients a_3 to a_8 with the result:

$$\sigma_{\theta\theta} = \pm\frac{4\mu r}{\pi R(\kappa+1)}[\ln r - const], \tag{A.4}$$

where the constant term arises because the other two coefficients are arbitrary. As $r = |x'|$ (see Fig. 2(b)), we can write:

$$p_{\text{asy}}(x') = -\frac{x'}{\pi RA}\big[\ln|x'| - const\big]. \tag{A.5}$$

The Airy stress function corresponding to the complete asymptotic stress field is

$$\Phi_{\text{asy}}(r,\theta)=\frac{\mu r^3}{R(\kappa+1)}\bigg[a_1\cos\theta+a_2\cos 3\theta-\sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{\pi}(\ln r\cos\theta-\theta\sin\theta)-\frac{1}{3\pi}(\ln r\cos 3\theta-\theta\sin 3\theta)\bigg], \tag{A.6}$$

where $const=\frac{3}{2}(a_1+a_2)$.

Appendix B. Corrective term: Green's function formulation

For the sharp corner, the logarithmic singularity has the form

$$p(x)=-\frac{2\phi}{\pi A}\ln|x|. \tag{B.1}$$

For the rounded problem (Fig. 2(c)), we replace the smooth profile by a series of discontinuities in surface gradient. Defining $\psi(x)$ as the change in slope per unit length, the contact pressure can then be found by superposition so that

$$p(x)=-\frac{2}{\pi A}\int_{x_1}^{x_2}\psi(\xi)\ln|x-\xi|\,d\xi. \tag{B.2}$$

In the special case under examination, where the curvature is constant, and the total change in slope is 2ϕ,

$$\psi(x)=\psi=\frac{\phi}{2b}. \tag{B.3}$$

It is indeed easy to demonstrate that we obtain the same asymptotic expression for the contact pressure distribution as in (17):

$$p^*(x)=-\frac{\phi}{\pi Ab}\int_{-b}^{b}\ln|x-\xi|\,d\xi=-\frac{2\phi}{\pi A}\bigg[\frac{1}{2}\bigg(1-\frac{x}{b}\bigg)\ln|b-x|+\frac{1}{2}\bigg(1+\frac{x}{b}\bigg)\ln|b+x|-1\bigg]. \tag{B.4}$$

Appendix C. Sharp nosed, rigid, finite indenter

The surface displacement gradient for the problem depicted in Fig. 1(c) is given by

$$\frac{dv}{dx}=v'(x)=\begin{cases}\phi, & -c<x<0,\\ -\phi, & 0<x<c\end{cases} \tag{C.1}$$

now require a 'singular both ends' form of solution. The integral equation connecting the contact pressure, $p(x)$, with the surface profile is

$$\frac{1}{A}\frac{\partial v}{\partial x}=-\frac{1}{\pi}\int_{-a}^{a}\frac{p(t)}{t-x}\,dt. \tag{C.2}$$

The contact pressure is singular as $x\to\pm c$, so the fundamental form of the solution obtained using the Riemann–Hilbert procedure is

$$p(x)=-\frac{w(x)}{\pi}\left(\int_{-c}^{c}\frac{v'(t)/A}{w(t)(t-x)}\,dt+\frac{2C\phi}{A}\right), \tag{C.3}$$

where C is a constant and

$$w(x)=\frac{1}{\sqrt{c^2-x^2}}. \tag{C.4}$$

Therefore

$$p(x) = -\frac{w(x)\phi}{\pi A}\left\{\int_{-c}^{0} \frac{1}{w(t)(t-x)}\,dt - \int_{0}^{c} \frac{-1}{w(t)(t-x)}\,dt + 2C\right\}. \tag{C.5}$$

Now

$$\int \frac{dt}{w(t)(t-x)} = \frac{F(c,t,x)}{w(x)} + \frac{1}{w(t)} - x\sin^{-1}\left(\frac{t}{c}\right), \tag{C.6}$$

where

$$F(c,t,x) = \ln\left|\frac{\sqrt{(c-x)/(c+x)} - \sqrt{(c-t)/(c+t)}}{\sqrt{(c-x)/(c+x)} + \sqrt{(c-t)/(c+t)}}\right| \tag{C.7}$$

and we note

$$F(c,\pm c,x) = 0. \tag{C.8}$$

Thus

$$\int_{-c}^{0} \frac{dt}{w(t)(t-x)} = \frac{F(c,0,x)}{w(x)} + c - x\frac{\pi}{2}, \tag{C.9}$$

$$\int_{0}^{c} \frac{dt}{w(t)(t-x)} = -\frac{F(c,0,x)}{w(x)} - c - x\pi, \tag{C.10}$$

i.e.

$$p(x) = -\frac{2\phi}{\pi A}\big(F(c,0,x) + (c+C)w(x)\big). \tag{C.11}$$

We can now find C by imposing vertical equilibrium

$$P = \int_{-c}^{c} p(x)\,dx, \tag{C.12}$$

i.e.

$$P = \frac{2C\phi}{A\pi}\left(\int_{-c}^{c} w(x)\,dx + \int_{-c}^{c} \frac{v'(t)}{w(t)}\left\{\int_{-c}^{c} \frac{w(x)}{t-x}\,dx\right\}\right)dt. \tag{C.13}$$

The inner integral here is a standard CPV integral with value zero. Therefore:

$$P = \frac{2C\phi}{A\pi}\int_{-c}^{c} \frac{1}{\sqrt{c^2-x^2}}\,dx = \frac{2C\phi}{A}. \tag{C.14}$$

Thus

$$p(x) = -\frac{2\phi}{\pi A}\left(F(c,0,x) + \left(c + \frac{AP}{2\phi}\right)\frac{1}{\sqrt{c^2-x^2}}\right). \tag{C.15}$$

Appendix D. Round nosed, rigid, finite indenter

The surface displacement gradient for this problem is given by

$$\frac{dv}{dx} = v'(x) = \begin{cases} \phi, & -a < x < -b, \\ -\dfrac{x}{R}, & -b < x < b, \\ -\phi, & b < x < a. \end{cases} \tag{D.1}$$

The integral equation connecting the surface profile with the contact pressure is given by Eqs. (C.3), (C.4). The overall normal equilibrium equation determines the value of C. The first term of the pressure in Eq. (C.3) is given by

$$p(x) = -\frac{w(x)}{\pi AR}\left\{cb\int_{-c}^{-b}\frac{dt}{w(t)(t-x)} - \int_{-b}^{b}\frac{dt}{w(t)} - x\int_{-b}^{b}\frac{dt}{w(t)(t-x)} - b\int_{b}^{c}\frac{dt}{w(t)(t-x)}\right\}. \tag{D.2}$$

Now we know that

$$\int\frac{dt}{\sqrt{c^2-t^2}(t-x)} = \frac{F(c,t,x)}{\sqrt{c^2-x^2}} \tag{D.3}$$

and from Appendix C

$$\int\frac{dt}{w(t)(t-x)} = \frac{F(c,t,x)}{w(x)} + \frac{1}{w(t)} - x\sin^{-1}\left(\frac{t}{c}\right). \tag{D.4}$$

Also

$$\int\sqrt{c^2-x^2}\,dx = \frac{c^2}{2}\sin^{-1}\left(\frac{t}{c}\right) + \frac{t\sqrt{c^2-t^2}}{2}. \tag{D.5}$$

So the pressure distribution becomes:

$$p(x) = -\frac{1}{\pi AR}\left\{(b-x)F(c,b,x) + (b+x)F(-b,c,x) + w(x)\left[b\sqrt{c^2-b^2} + \left(2x^2-c^2\right)\sin^{-1}\left(\frac{b}{c}\right)\right] + \frac{w(x)2CR}{\phi}\right\}. \tag{D.6}$$

Now, for vertical equilibrium

$$P = \int_{-c}^{c} p(x)\,dx, \tag{D.7}$$

i.e.

$$P = \int_{-c}^{c}\left\{\frac{w(x)}{\pi}\int_{-c}^{c}\frac{v'(t)/A}{w(t)(t-x)}\,dt + Cw(x)\right\}dx = \frac{1}{A\pi}\int_{-c}^{c}\frac{v'(t)}{w(t)}\left\{\int_{-c}^{c}\frac{w(x)}{t-x}\,dx\right\}dt + C\int_{-c}^{c}w(x)\,dx. \tag{D.8}$$

The inner integral in the first term here is a standard CPV integral with value zero.

So

$$P = C\int_{-c}^{c}w(x)\,dx = \pi C. \tag{D.9}$$

References

Barber, J.R., 2002. Elasticity, second ed., Kluwer Academic Publishers, Dordrecht.

Churchman, C.M., Sackfield, A., Hills, D.A., 2006. Asymptotic solutions for contact problems: the effect of an internal discontinuity in surface profile. Proceedings of the Institution of Mechanical Engineers Part C – Journal of Mechanical Engineering Science 220 (4), 387–391.

Ciavarella, M., Hills, D.A., Monno, G., 1998. Contact problems for a wedge with rounded apex. International Journal of Mechanical Sciences 40 (10), 977–988.

Dini, D., Hills, D.A., 2003. A method based on asymptotics for the refined solution of almost complete partial slip contact problems. European Journal of Mechanics A/Solids 22 (6), 851–859.

Dini, D., Hills, D.A., 2004. Bounded asymptotic solutions for incomplete contacts in partial slip. International Journal of Solids and Structures 41 (24–25), 7049–7062.

Hills, D.A., Dini, D., Mugadu, A., Korsunsky, A.M., 2004. A review of asymptotic procedures in stress analysis: known solutions and their applications. Journal of Strain Analysis for Engineering Design 39 (6), 553–568.

Sackfield, A., Dini, D., Hills, D.A., 2005. The tilted shallow wedge problem. European Journal of Mechanics A/Solids 24 (6), 919–928.

Sackfield, A., Dini, D., Hills, D.A., 2006. The contact problem for a wheel having a 'flat'. Wear 261 (11–12), 1265–1270.
Sackfield, A., Dini, D., Hills, D.A., 2007. Contact of a rotating wheel with a flat. International Journal of Solids and Structures 44 (10), 3304–3316.
Truman, C.E., Sackfield, A., Hills, D.A., 1995. Contact mechanics of wedge and cone indenters. International Journal of Mechanical Sciences 37 (3), 261–275.