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Indentation of the semi-infinite elastic solid by a concave rigid punch

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ABSTRACT

A solution is obtained for the relationship between load, displacement and inner contact radius for an axisymmetric, spherically concave, rigid punch, indenting an elastic half-space. Analytic approximations are developed for the limiting cases in which the ratio of the inner and outer radii of the annular contact region is respectively small and close to unity. These approximations overlap well at intermediate values. The same method is applied to the conically concave punch and to a punch with a central hole.

АБСТРАКТ

Получено решение для взаимоотношения между загрузкой, смещением и внутренним радиусом контакта для аксисимметричного сферично вогнутого жесткого пробойника, вдавливающегося на упругом полупространстве.

Аналитические приближения развиваются для предельних случаев, когда отношение внутренних и внешних радиусов кольцевого района контакта или мало или близко к единице. Эти приближения хорошо совпадают при промежуточных значениях.

Этот метод применяется к коническому вогнутому пробойнику и к пробойнику с центральным отверстием.

1. Introduction

This paper is concerned with the indentation of a semi-infinite elastic solid by an axisymmetric rigid punch, the face of which is slightly concave (see fig. 1). When the indenting load P is small, the contact area will be a thin annulus, the outer circumference of which coincides with the edge of the punch. The inner circumference will shrink with increasing load and there will be a critical load above which the entire punch face makes contact with the half-space. The contact problem for high loads can therefore be treated by classical methods (e.g., [1]) and we can obtain an expression for the critical value of load by applying the condition that contact stress must be everywhere compressive.

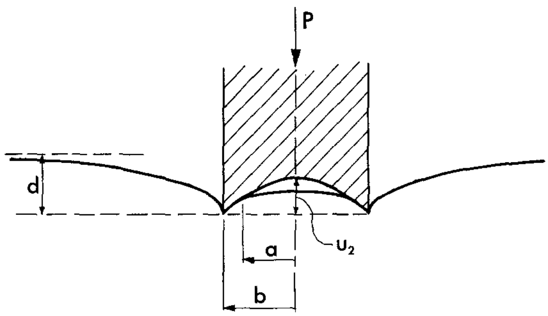

Figure 1. Indentation of the semi-infinite solid by a spherically concave punch.

There remains the more interesting case in which the load is less than this value and the contact region is annular.

Problems involving annular contact regions have attracted considerable attention in recent years and several solutions have been given for the case in which the punch face is flat, all of which can be readily extended to more complex punch profiles. Most of these solutions are based on a perturbation of that for a complete circular punch and are therefore developed in terms of an expansion in powers of the ratio of inner to outer radius. One of the most elegant solutions of this kind is that due to Collins [2] and more detail of this type of solution is given by Jain and Kanwal [3].

An alternative method, appropriate to the case where this ratio approaches unity, is to perturb the corresponding two-dimensional indentation problem. In this method, first used by Grinberg and Kuritsyn [4], the "small parameter" is the ratio between the thickness of the annulus and its mean diameter. The pressure distribution and punch profile are represented by series, the coefficients of which are connected by a series of simultaneous equations or an integral equation. A more recent numerical solution by Shibuya, Koizumi and Nakahara [5] also uses this method of representation, though without specifying that the thickness of the annulus should be small.

The problem of the concave punch differs from annular contact problems so far treated in that the inner radius of the annulus is not known *a priori*, but has to be found as part of the solution. It can be determined from the condition that contact stress is bounded at the inner edge of the contact region, though sometimes it proves convenient to use an equivalent variational statement.

We restrict our attention to the determination of the relationship between load, contact radius and normal displacement, since these are the quantities of greatest physical interest and once they are known the solution can be completed by existing methods.

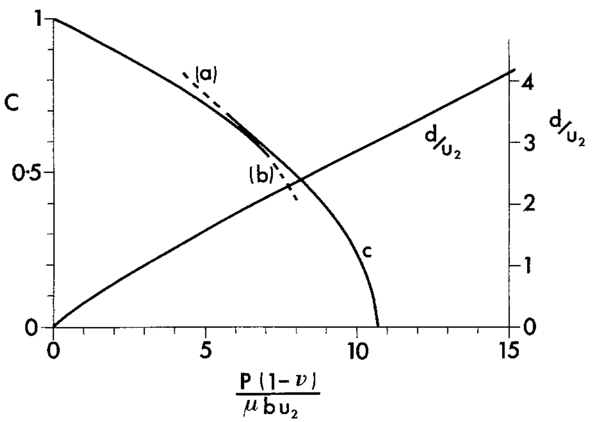

Figure 2. The ratio of contact radii, c, and the non-dimensional displacement, d/u_2 as functions of non-dimensional load, $P(1-\nu)/\mu b u_2$ for the spherically concave punch. Curves (a, b) are derived from equations (12, 25) respectively.

The cases in which the ratio of radii is respectively small and close to unity are treated in sections 2 and 3 and the results are presented in fig. 2. It is found that a fairly modest degree of approximation gives a satisfactory overlap of these solutions at intermediate values. In sections 4, 5, corresponding results are presented for two other punch profiles.

2. Ratio of contact radii small

In this section, we shall adapt the solution due to Collins [2] to the case in which the inner radius of the contact region is unknown. In order to find this radius, we make use of a variational formulation of the contact condition discussed in a previous paper [6]. We define the function $P(A)$ as the total load on the punch required to produce the prescribed displacement profile over the region A without regard to the sign of the "contact" stress. It can then be shown that the maximum value of $P(A)$ occurs when A is chosen in such a way that (i) there are no regions of tensile contact stress within A; (ii) outside A, the displacement is everywhere greater than or equal to the prescribed value (i.e., there is clearance between the punch and the half-space). These are precisely the conditions which determine the extent of the contact area in a practical indentation problem.

We can therefore find the inner contact radius (a) for the concave punch by finding P for arbitrary a and applying the condition $\partial P/\partial a = 0$. This method requires that an analytic expression be found for P, but it does not require a complete solution for the contact stress distribution.

We denote the inner and outer radii of the contact region by a, b respectively and write $c = a/b$. The prescribed normal displacement $u(r)$ under the punch is decomposed into two components: $f_0(r)$ which is bounded at the origin and $f_1(r)$ which tends to zero at infinity.

With this notation, Collins [2] shows that the total load on the punch can be expressed in the form

$$\frac{P(1-\nu)}{2\pi\mu} = \int_0^b g_0(t)\mathrm{d}t + \int_b^\infty g_2(t)\mathrm{d}t - \frac{1}{\pi}\int_0^a g(t)\log\left(\frac{b-t}{b+t}\right)\mathrm{d}t \tag{1}$$

where μ, ν are respectively the modulus of rigidity and Poisson's ratio for the material,

$$g_0(t) = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\omega f_0(\omega)\mathrm{d}\omega}{(t^2-\omega^2)^{\frac{1}{2}}}, \qquad (0 \leqslant t < b) \tag{2}$$

$$g_2(t) = \frac{4}{\pi^2}\int_t^\infty (\omega^2-t^2)^{-\frac{1}{2}}\left\{\frac{\mathrm{d}}{\mathrm{d}\omega}\int_a^\omega \left[\frac{\mathrm{d}}{\mathrm{d}s}\int_s^\infty \frac{rf_1(r)\mathrm{d}r}{(r^2-s^2)^{\frac{1}{2}}}\right]\frac{s\mathrm{d}s}{(\omega^2-s^2)^{\frac{1}{2}}}\right\}\mathrm{d}\omega\,, \qquad (b<t<\infty) \tag{3}$$

and $g(t)$ is the solution of the Fredholm equation

$$g(t) + \frac{1}{\pi^2}\int_0^a \frac{g(s)}{(t^2-s^2)}\left[2s\log\left(\frac{b-t}{b+t}\right) - 2t\log\left(\frac{b-s}{b+s}\right)\right]\mathrm{d}s$$
$$= \frac{2}{\pi}\int_0^t (t^2-\omega^2)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\omega}\int_\omega^b \frac{sg_0(s)\mathrm{d}s}{(s^2-\omega^2)^{\frac{1}{2}}}\mathrm{d}\omega - \frac{2t}{\pi}\int_b^\infty \frac{g_2(s)\mathrm{d}s}{(t^2-s^2)}\cdot \qquad (0 \leqslant t < a) \tag{4}$$

An iterative solution to equation (4) can be obtained if $c\,(=a/b)$ is small, and substitution into equation (1) gives an analytic approximation for P as a function of c.

For the spherically concave punch of fig. 1, the prescribed displacement is

$$u(r) = (d-u_2)+u_2(r/b)^2 \tag{5}$$
$$= f_0(r)\,;$$
$$f_1(r) = 0\,,$$

where d is the displacement from the position of zero load and u_2 is a constant representing the concavity of the punch.

It is convenient to treat the two terms of this expression separately and superpose the results.

The first term corresponds to a constant normal displacement under the punch of $(d-u_2)$ and has already been treated by Collins [2] and Jain and Kanwal [3] who give the result

$$P(c) = \frac{4\mu b(d-u_2)}{(1-\nu)}\left\{1 - \frac{4c^3}{3\pi^2} - \frac{8c^5}{15\pi^2} - \frac{16c^6}{27\pi^4} - \frac{92c^7}{315\pi^2} - \frac{448c^8}{675\pi^4} + O(c^9)\right\}. \tag{6}$$

For the second term, we substitute $f_0(r) = u_2(r/b)^2$ into equations (2, 3) to obtain

$$g_0(t) = \frac{4u_2t^2}{\pi b^2}\,; \qquad g_2(t) = 0\,, \tag{7}$$

in which case the right hand side of equation (4) is

$$\frac{8u_2}{\pi^2b^2}\int_0^t (t^2-\omega^2)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\omega}\int_\omega^b \frac{s^3\mathrm{d}s}{(s^2-\omega^2)^{\frac{1}{2}}}\,\mathrm{d}\omega = \frac{8t^2u_2}{\pi^2b^2}\left\{\frac{1}{2}\log\left(\frac{b-t}{b+t}\right) + \frac{b}{t}\right\}. \tag{8}$$

An approximate solution to equation (4) can now be obtained by the usual iterative method and is

$$\frac{g(t)}{u_2} = \frac{8t}{\pi^2 b} - \frac{8t^3}{\pi^2 b^3} - \frac{32c^3 t}{9\pi^4 b} - \frac{8t^5}{3\pi^2 b^5} - \frac{64c^5 t}{75\pi^4 b} + \frac{32c^3 t^3}{15\pi^4 b^3}$$
$$- \frac{8t^7}{5\pi^2 b^7} + \frac{128c^5 t^3}{135\pi^6 b^3} + O(c^9) \qquad (0 \leqslant t < a). \tag{9}$$

Finally, substituting into equation (1) we obtain

$$P(c) = \frac{8\mu b u_2}{3(1-\nu)}\left\{1 + \frac{4c^3}{\pi^2} - \frac{8c^5}{5\pi^2} - \frac{16c^6}{9\pi^4} - \frac{4c^7}{5\pi^2} - \frac{32c^8}{225\pi^4} + O(c^9)\right\}. \tag{10}$$

We now superpose the results of equations (6, 10) to obtain

$$P(c) = \frac{4\mu b}{(1-\nu)}\left\{(d-u_2)\left[1 - \frac{4c^3}{3\pi^2} - \frac{8c^5}{15\pi^2} - \frac{16c^6}{27\pi^4} - \frac{92c^7}{315\pi^2} - \frac{448c^8}{675\pi^4} + O(c^9)\right]\right.$$
$$\left. + \frac{2u_2}{3}\left[1 + \frac{4c^3}{\pi^2} - \frac{8c^5}{5\pi^2} - \frac{16c^6}{9\pi^4} - \frac{4c^7}{5\pi^2} - \frac{32c^8}{225\pi^4} + O(c^9)\right]\right\}, \tag{11}$$

for the total load on the punch shown in fig. 1.

To find the unknown ratio c, we apply the condition that this load must be maximum i.e., $\partial P(c)/\partial c = 0$ and hence

$$\frac{d}{u_2} = 1 + 2\left\{\frac{1 - \dfrac{2c^2}{3} - \dfrac{8c^3}{9\pi^2} - \dfrac{7c^4}{15} - \dfrac{64c^5}{675\pi^2} + O(c^6)}{1 + \dfrac{2c^2}{3} + \dfrac{8c^3}{9\pi^2} + \dfrac{23c^4}{45} + \dfrac{896c^5}{675\pi^2} + O(c^6)}\right\}. \tag{12}$$

The relation between P and c can be found by substituting for d from equation (12) into equation (11).

These results are plotted in fig. 2 as curve (a). The inner radius of the contact area falls with increasing load as expected and reaches zero at the critical value

$$P = \frac{32\mu b u_2}{3(1-\nu)}, \tag{13}$$

when the displacement

$$d = 3u_2\,. \tag{14}$$

For higher values of load, the whole face of the punch will make contact with the half-space and the contact stress distribution can be written down by superposing the solutions for a flat punch and a parabolic punch; *i.e.*,

$$\sigma_{zz} = \frac{-2\mu u_2}{\pi(1-\nu)b}\left\{(1+d/u_2)(1-r^2/b^2)^{-\frac{1}{2}} - 4(1-r^2/b^2)^{\frac{1}{2}}\right\}, \tag{15}$$

corresponding to a total load

$$P = \frac{4\mu b u_2}{(1-\nu)}\{d/u_2 - \tfrac{1}{3}\}\,. \tag{16}$$

Equation (15) will define regions of tensile stress near $r=0$ unless $d>3u_2$, agreeing with equation (14). The limiting value of load can also be found by substituting into equation (16) and the result agrees with that obtained from the annular solution as equation (13). The high load solution (equation (16)) is shown as the continuation in $P>32\mu b u_2/3(1-\nu)$ in fig. 1.

3. Ratio of contact radii near unity

We now consider the case in which c approaches unity, making use of the method of Grinberg and Kuritsyn [4].

We define the quantities

$$\left.\begin{aligned} &R=\tfrac{1}{2}(a+b)\,; \quad h=\tfrac{1}{2}(b-a)\,; \\ &\operatorname{Cos}\alpha=(r-R)/h\,; \\ &\varepsilon=h/R=\left(\frac{1-c}{1+c}\right), \end{aligned}\right\} \tag{17}$$

where ε is small and α takes values between $-\pi$, $+\pi$ in the contact region.

The prescribed displacement profile is expanded as a series of the form

$$u(\varepsilon,\alpha) = \frac{U_0^0}{2} + \varepsilon\{U_1^1\operatorname{Cos}\alpha\} + \varepsilon^2\left\{\frac{U_2^0}{2} + U_2^2\operatorname{Cos}2\alpha\right\} + \varepsilon^3\{U_3^1\operatorname{Cos}\alpha + U_3^3\operatorname{Cos}3\alpha\} + \ldots \tag{18}$$

and it is assumed that the appropriate contact stress distribution can be represented in the form

$$\sigma_{zz}(\alpha) = \frac{-2\pi\mu v(\varepsilon,\alpha)}{(1-\nu)R\varepsilon(1+\varepsilon\operatorname{Cos}\alpha)\operatorname{Sin}\alpha} \tag{19}$$

where $v(\varepsilon, \alpha)$ can be expanded as

$$v(\varepsilon,\alpha) = \sum_{n=0}^{\infty} V_n \varepsilon^n\,.$$

The V_n are functions of α which also depend on ε, but only logarithmically.

The coefficients V_n can then be found up to any desired degree of accuracy by an iterative solution of the appropriate integral equation with ε as parameter.

Grinberg and Kuritsyn give the first four coefficients as

$$\left.\begin{aligned}
V_0 &= U_0^0/4\pi L;\\
V_1 &= \frac{(U_0^0+4U_1^1)\,\mathrm{Cos}\,\alpha}{8\pi};\\
V_2 &= \frac{1}{64\pi}\{[2U_0^0+8U_1^1-5U_0^0/L+16U_2^0/L+4U_0^0/L^2]\\
&\qquad +[U_0^0+24U_1^1+64U_2^2-3U_0^0/2L]\,\mathrm{Cos}\,2\alpha\}\quad;\\
V_3 &= \frac{1}{512\pi}\{[-4L(U_0^0+4U_1^1)+9U_0^0+20U_1^1+64U_2^0\\
&\qquad +192U_2^2+256U_3^1-6U_0^0/L]\,\mathrm{Cos}\,\alpha\\
&\qquad +[4U_1^1-3U_0^0+320U_2^2+768U_3^3+6U_0^0/L]\,\mathrm{Cos}\,3\alpha\}\,,
\end{aligned}\right\}\tag{21}$$

where $L=\log(16/\varepsilon)$.

Substituting for r, b into the equation (5) of the profile of the spherically concave punch, we obtain

$$u(\alpha)=\left\{d+u_2\left(\frac{1}{(1+\varepsilon)^2}-1\right)\right\}+\frac{2u_2\varepsilon\,\mathrm{Cos}\,\alpha}{(1+\varepsilon)^2}+\frac{u_2\varepsilon^2(1+\mathrm{Cos}\,2\alpha)}{2(1+\varepsilon)^2},\tag{22}$$

corresponding to equation (18) with non-zero coefficients

$$\left.\begin{aligned}
U_0^0 &= 2\left\{d+u_2\left(\frac{1}{(1+\varepsilon)^2}-1\right)\right\};\\
U_1^1 &= 2u_2/(1+\varepsilon)^2\,;\\
U_2^0 &= u_2/(1+\varepsilon)^2\,;\\
U_2^2 &= u_2/2(1+\varepsilon)^2\,.
\end{aligned}\right\}\tag{23}$$

We can now substitute into equations (19–21) to obtain an expression for the contact stress (which is omitted here in the interests of brevity).

The unknown ratio ε can be found from the condition that this expression is bounded at $\alpha=\pi$ and hence, from equation (19),

$$v(\varepsilon,\pi)=0\,.\tag{24}$$

This condition leads to the result

$$\frac{d}{u_2}=1-\frac{\left\{\dfrac{1}{L}-\dfrac{5\varepsilon}{2}+\dfrac{51\varepsilon^3}{16}+\dfrac{3\varepsilon^2}{32L}+\dfrac{\varepsilon^2}{4L^2}+\dfrac{5\varepsilon^3L}{32}-\dfrac{55\varepsilon^3}{64}+\dots\right\}}{\{1+\varepsilon\}^2\left\{\dfrac{1}{L}-\dfrac{\varepsilon}{2}+\dfrac{3\varepsilon^2}{16}-\dfrac{13\varepsilon^2}{32L}+\dfrac{\varepsilon^2}{4L^2}+\dfrac{\varepsilon^3L}{32}-\dfrac{3\varepsilon^3}{64}+\dots\right\}},\tag{25}$$

whilst the total load, obtained by integrating the contact stress over the contact area, is

$$P=\frac{4\pi^2\mu b u_2}{(1-\nu)(1+\varepsilon)^3}\left\{\frac{\varepsilon^2}{4L}+\frac{\varepsilon^2}{4}+\left[\left(\frac{d}{u_2}-1\right)(1+\varepsilon)^2+1\right]\left[\frac{1}{2L}+\frac{\varepsilon^2}{32}\left(2-\frac{5}{L}+\frac{4}{L^2}\right)\right]\right\}.\tag{26}$$

These results are shown as curve (b) in fig. 2. A good overlap is obtained between curves (a, b), thus providing some measure of the degree of approximation involved as well as a useful check on the calculations.

The relationships between load and displacement obtained from the two methods agree so closely that they are shown on fig. 2 as a single curve, continuous with equation (16) at $c=0$.

4. The conical punch

As a second example of the method, we shall consider the punch shown in fig. 3(a), the face of which has a slight conical depression.

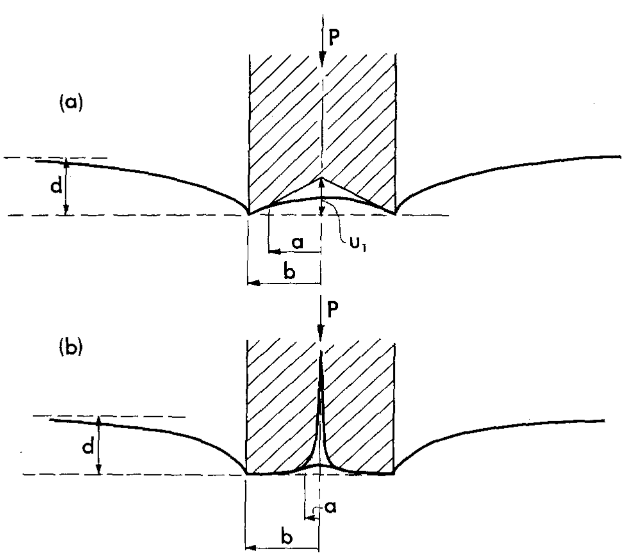

Figure 3. Indentation of the semi-infinite solid by (a) a conically concave punch; (b) an inverse power punch.

The prescribed displacement is

$$\begin{aligned} u(r) &= (d-u_1)+u_1(r/b) \\ &= f_0(r)\,; \\ f_1(r) &= 0\,. \end{aligned} \tag{27}$$

Substituting into equations (1–4) of section 2 and making use of equation (6) for the first (constant) term we obtain

$$\begin{aligned} P(c) = \frac{4\mu b}{(1-\nu)}\Bigg\{(d-u_1)\left(1-\frac{4c^3}{3\pi^2}-\frac{8c^5}{15\pi^2}-\frac{16c^6}{27\pi^4}-\frac{92c^7}{315\pi^2}-\frac{448c^8}{675\pi^4}+O(c^9)\right) \\ +\frac{\pi u_1}{4}\left(\left[1+\frac{8c^3}{9\pi^2}-\frac{52c^5}{75\pi^2}-\frac{32c^6}{81\pi^4}-\frac{326c^7}{735\pi^2}+\frac{496c^8}{10125\pi^4}+O(c^9)\right]\right. \\ \left.-\frac{8c^3\log c}{3\pi^2}\left[1+\frac{c^2}{5}-\frac{4c^3}{9\pi^2}+\frac{3c^4}{35}-\frac{92c^5}{45\pi^2}+O(c^6)\right]\right)\Bigg\} \end{aligned} \tag{28}$$

as an approximation for small values of c.

The condition $\partial P(c)/\partial c = 0$ gives

$$\frac{d}{u_1} = 1 - \frac{\pi}{4}\left\{\frac{c^2 + \dfrac{16c^3}{81\pi^2} + \dfrac{5c^4}{6} - \dfrac{14792c^5}{10125\pi^2} + \log c\left(2 + \dfrac{2c^2}{3} - \dfrac{16c^3}{9\pi^2} + \dfrac{2c^4}{5} - \dfrac{1472c^5}{135\pi^2}\right)}{1 + \dfrac{2c^2}{3} + \dfrac{8c^3}{9\pi^2} + \dfrac{23c^4}{45} + \dfrac{896c^5}{675\pi^2}}\right\} \quad (29)$$

to determine the unknown ratio c.

To find an approximate solution for values of c near unity, we express equation (27) as a function of α in the form

$$u(\alpha) = \left\{d - \frac{\varepsilon u_1}{(1+\varepsilon)}\right\} + \frac{u_1 \varepsilon \operatorname{Cos} \alpha}{(1+\varepsilon)}, \quad (30)$$

making use of equation (17). Thus, the non-zero coefficients in equation (18) are

$$\left.\begin{aligned} U_0^0 &= 2\left(d - \frac{\varepsilon u_1}{(1+\varepsilon)}\right), \\ U_1^1 &= u_1/(1+\varepsilon). \end{aligned}\right\} \quad (31)$$

Substituting into equations (18–20) and integrating over the contact area, we obtain

$$P = \frac{4\pi^2 \mu b u_1}{(1-\nu)(1+\varepsilon)^2}\left\{\frac{\varepsilon^2}{8} + \left[(1+\varepsilon)\frac{d}{u_1} - \varepsilon\right]\left[\frac{1}{2L} + \frac{\varepsilon^2}{32}\left(2 - \frac{5}{L} + \frac{4}{L^2}\right)\right]\right\}. \quad (32)$$

The continuity condition at $\alpha = \pi$ gives

$$\frac{d}{u_1} = 1 - \frac{\left\{\dfrac{1}{L} - \dfrac{3\varepsilon}{2} + \dfrac{\varepsilon^2}{16}\left(19 - \dfrac{13}{2L} + \dfrac{4}{L^2}\right) - \dfrac{\varepsilon^3}{64}(9 - 6L)\right\}}{(1+\varepsilon)\left\{\dfrac{1}{L} - \dfrac{\varepsilon}{2} + \dfrac{\varepsilon^2}{16}\left(3 - \dfrac{13}{2L} + \dfrac{4}{L^2}\right) - \dfrac{\varepsilon^3}{64}(3 - 2L)\right\}}. \quad (33)$$

These relations between P, c, d are shown graphically in fig. 4, curve (a) being the approximation for small values of c (equations (28, 29) and curve (b) that for values near unity (equations (32, 33)). As before, a good overlap is obtained between the two methods at intermediate values.

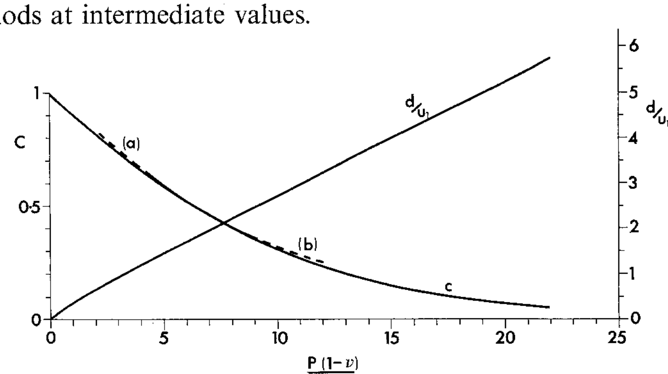

Figure 4. The ratio of contact radii, c, and the non-dimensional displacement, d/u_1, as functions of non-dimensional load, $P(1-\nu)/\mu b u_1$, for the conically concave punch. Curves (a, b) are derived from equations (29, 33) respectively.

There is no critical load in the case of the concave conical punch; the ratio c continues to fall with increasing load, but never reaches zero. This is because any solution with finite total load involving complete contact over the punch face would have an unacceptable tensile contact stress singularity at the origin. When the load is high, we have approximately

$$c=\exp\left(-\frac{2}{\pi}\left\{\frac{d}{u_1}-1\right\}\right)=\exp\left(\frac{1}{2}-\frac{P(1-\nu)}{2\pi\mu b u_1}\right). \tag{34}$$

5. The inverse power punch

Finally we consider the punch shown in fig. 3(b) for which the prescribed displacement is

$$u(r)=(d+u_{-2})-u_{-2}(b/r)^2\,. \tag{35}$$

This problem is one in which $f_1(r)$ (equation (3)) is non-zero and it is also of interest insofar as it models the punch with a smooth curvature into a small central hole. For brevity, we shall restrict our attention to the solution for small values of c.

The first (constant) term is treated as in the previous two examples. For the second term, the functions $f_0(t)$, $g_0(t)$ are zero,

$$f_1(t)=-u_{-2}(b/t)^2 \tag{36}$$

and

$$g_2(t)=-\frac{u_{-2}b^2}{\pi t^2}\log\left(\frac{t^2}{a^2}-1\right), \tag{37}$$

from equation (3).

Substitution into equations (4, 1) respectively gives an expression for total load associated with the second term which, when the contribution from the first term is added in, gives

$$P(c)=\frac{4\mu b}{(1-\nu)}\left\{(d+u_{-2})\left(1-\frac{4c^3}{3\pi^2}-\frac{8c^5}{15\pi^2}-\frac{16c^6}{27\pi^4}-\frac{92c^7}{315\pi^2}-\frac{448c^8}{675\pi^4}+O(c^9)\right)+\right.$$
$$+\frac{u_{-2}}{2}\left(-2+\frac{c^2}{3}-\frac{8c^3}{27\pi^2}+\frac{c^4}{10}+\frac{28c^5}{1125\pi^2}+\frac{32c^6}{243\pi^4}+\frac{c^6}{21}+\right.$$
$$\left.\left.+\log c\left[2+\frac{8c^3}{9\pi^2}+\frac{64c^5}{75\pi^2}-\frac{32c^6}{87\pi^4}\right]\right)\right\}. \tag{38}$$

The continuity condition $\partial P(c)/\partial c=0$ gives

$$\frac{d}{u_{-2}}=-1+\frac{\pi^2}{4c^3}\left\{\frac{1+\frac{c^2}{3}+\frac{c^4}{5}+\frac{22c^5}{45\pi^2}-\frac{32c^6}{81\pi^4}+\frac{c^6}{7}+\frac{4c^3\log c}{3\pi^2}\left(1+\frac{8c^2}{5}-\frac{8c^3}{9\pi^2}\right)}{1+\frac{2c^2}{3}+\frac{8c^3}{9\pi^2}+\frac{23c^4}{45}+\frac{896c^5}{675\pi^2}}\right\} \tag{39}$$

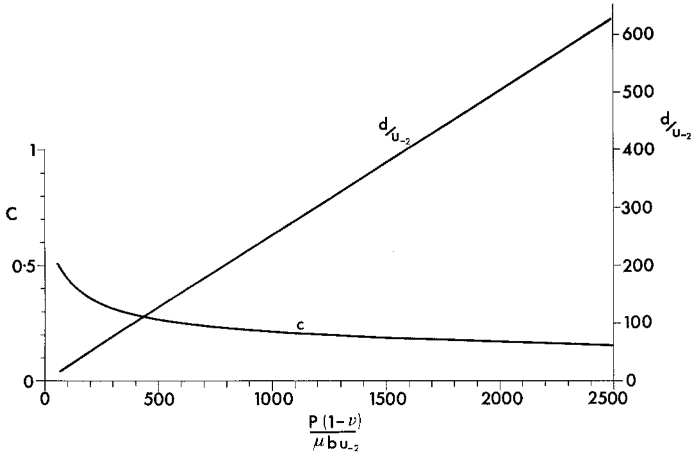

Figure 5. The ratio of contact radii, c, and the non-dimensional displacement, d/u_{-2} as functions of non-dimensional load, $P(1-\nu)/\mu bu_{-2}$ for the inverse power punch.

These results are shown graphically in fig. 5. As we should expect, c approaches zero asymptotically at large loads, under which conditions the approximate result

$$c = \left(\frac{\pi^2 u_{-2}}{4(d+u_{-2})}\right)^{\frac{1}{2}} = \left(\frac{\pi^2 \mu b u_{-2}}{P(1-\nu)}\right)^{\frac{1}{2}} \tag{40}$$

can be used.

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