J. R. Barber

Department of Mechanical Engineering,
University of Newcastle-upon-Tyne,
Newcastle-upon-Tyne NE1 7RU, England

Stability of Thermoelastic Contact for the Aldo Model

A perturbation method is used to investigate the stability of a simple one-dimensional rod model of thermoelastic contact which exhibits multiple steady-state solutions. A thermal contact resistance is postulated which is a continuous function of the contact pressure or separation. It is found that solutions involving substantial separation and/or contact pressures are always stable, but these are separated by unstable "imperfect contact" solutions in which one of the rods is very lightly loaded or has a very small separation. The results can be expressed in terms of the minimization of a certain energy function.

Introduction

A number of recent treatments of thermoelastic contact problems [1–3] have demonstrated that steady-state solutions are not necessarily unique if the hotter body has the lower thermal distortivity δ, defined by

$$\delta = \alpha(1 + \nu)/K \tag{1}$$

where α, ν, K are, respectively, the coefficient of linear thermal expansion, Poisson's ratio, and thermal conductivity.

In such cases, it is possible that some of the competing solutions are unstable, but if more than one are stable, the situation realized in practice will depend upon the history of heating and loading.

In an attempt to probe this question, Comninou and Dundurs [3] have considered a simplified thermoelastic contact system which they call the "Aldo Model." The three-dimensional contacting bodies are replaced by a large number of thin rods arranged normally to the interface and with frictionless and thermally insulated sides. This essentially constrains heat flow and load transfer to the normal direction.

Although this system is very much simpler than a real contact situation, it is sufficiently realistic to permit multiple solutions for the appropriate heat flow direction. Comninou and Dundurs have computed the total mechanical energies for these solutions, but these cannot be used to draw rigorous conclusions about stability, since the system is inherently nonconservative.

In this paper, the stability of steady-state solutions for the Aldo model will be investigated by an analysis of small transient perturbations. This method has already been successfully applied to the simpler problem of the one-dimensional rod confined between rigid walls at different temperatures [1] and leads to rigorous conclusions about stability which can be formulated in terms of an energy function.

Description of the Model

Comninou and Dundurs show that the cross section of the Aldo rods and the distribution of contacting and noncontacting rods over the interface do not influence the permissible steady-state solutions. In effect, a group of rods all in a similar state behaves as a single rod of proportionately greater cross-sectional area.

The essence of the Aldo model is therefore preserved if we consider the stability of a system of two rods of different cross-sectional areas A_1, A_2 as shown in Fig. 1.

The rods, both of length l, are rigidly joined at the top, where the temperature is maintained at zero. The other ends make contact with a rigid, perfectly conducting half space[1] at temperature T_0 (>0). The system is constrained so that only vertical displacements are permitted and a compressive contact force F is applied as shown.

As in the previous paper [1], we postulate the existance of a thermal contact reistance R_i (p_i, g_i) (i = 1, 2 for rods 1, 2, respectively) which depends on the pressure p_i between the rod and the half space, or on the gap g_i if the rod is not in contact. The stability of the system is not affected by the precise nature of this resistance function.

Steady-State Solution

Writing Q_i for the steady-state heat flux along rod i, and T_i for the temperature at the hot end, we have

$$Q_i = \frac{(T_0 - T_i)}{R_i} = \frac{A_i K\, T_i}{l} \tag{2}$$

and hence,

$$T_i = \frac{T_0}{(1 + A_i\, KR_i/l)} \tag{3}$$

[1] Comninou and Dundurs treat the contact of two systems of rods of different materials, but it is not anticipated that this more general case will introduce any qualitatively new features.

Contributed by the Applied Mechanics Division for publication in the JOURNAL OF APPLIED MECHANICS.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until December 1, 1981. Readers who need more time to prepare a discussion should request an extension from the Editorial Department. Manuscript received by ASME Applied Mechanics Division, July, 1980; final revision, January, 1981.

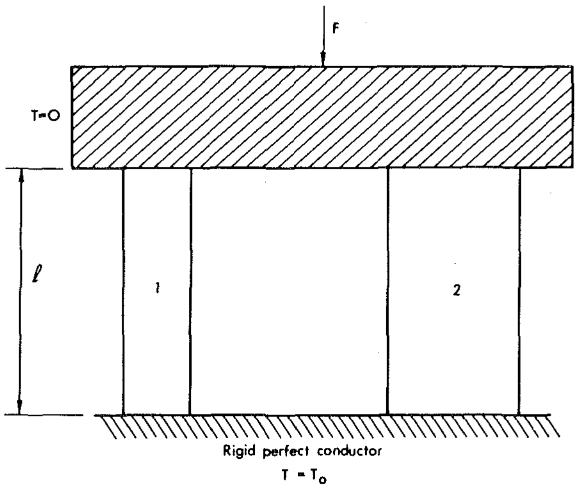

Fig. 1 The Aldo model

The unconstrained thermal expansion of the rod is therefore

$$u_i = \tfrac{1}{2}\alpha\, l\, T_i = u_0 f_i \qquad (4)$$

where $u_0 = \frac{1}{2}\alpha\, l\, T_0$ is the thermal expansion which would be developed if there were perfect thermal contact between the rod and the plane and

$$f_i = \frac{l}{(l + A_i K\, R_i)} \qquad (5)$$

This function tends to zero when the gap g_i is large ($R_i \to \infty$) and to unity when the contact pressure p_i is large ($R_i \to 0$) as shown in Fig. 2. In general, ,the transition between these limits would be expected to occur over a relatively small range of g_i, p_i.

Three possible contact states for the system can be distinguished as follows:

(*i*) **Rod 1 in Contact**

$$p_2 = 0, \quad p_1 = F/A_1,$$
$$g_1 = 0, \quad g_2 > 0,$$

and

$$g_2 = u_0(f_1 - f_2) - Fl/A_1E \qquad (6)$$

(*ii*) **Both Rods in Contact**

$$p_1, p_2 \geqslant 0,$$
$$g_1 = g_2 = 0,$$

and

$$u_0(f_1 - f_2) = (p_1 - p_2)l/E \qquad (7)$$

(*iii*) **Rod 2 in Contact**

$$p_1 = 0, \quad p_2 = F/A_2,$$
$$g_2 = 0, \quad g_1 > 0,$$

and

$$g_1 = u_0(f_2 - f_1) - Fl/A_2E \qquad (8)$$

We now define a piecewise continuous function x by the relations

$$\begin{aligned} x &= g_2 + Fl/A_1E; & g_2 &> 0, & (9i)\\ &= (p_1 - p_2)l/E; & g_1 &= g_2 = 0, & (9ii)\\ &= -g_1 - Fl/A_2E; & g_1 &> 0 & (9iii)\end{aligned}$$

In effect, x is the difference between the unconstrained thermal expansion of the two rods ($u_1 - u_2$). It is easily verified that this function is continuous at the transitions between the foregoing contact regimes. Furthermore, if equations (9*i*)–(9*iii*) are substituted into equations (6)–(8), respectively, the latter are all reduced to the form

$$(f_1 - f_2) = x/u_0 \qquad (10)$$

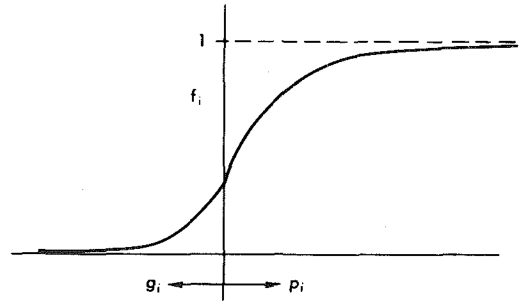

Fig. 2 Dependence of the function f_i on pressure (p_i) and gap (g_i)

Stability Analysis

In order to investigate the stability of the various solutions of equation (10), we examine the conditions under which a small perturbation from the steady state can grow exponentially with time.

The perturbation in temperature in the rods, ΔT_i, must satisfy the transient heat-conduction equation and the boundary condition

$$\Delta T_i(y) = 0 \quad \text{at} \quad y = 0 \qquad (11)$$

where y is measured from the cold end. The appropriate exponentially growing solution is

$$\Delta T_i(y) = B_i\, e^{at} \sinh \lambda y \qquad (12)$$

(see reference [1]), where

$$\lambda = (a/k)^{1/2} \qquad (13)$$

a, B_i are constants, t is time, and k is the thermal diffusivity of the material.

The corresponding perturbation in heat input to the rod is

$$\Delta Q_i = -A_i k \frac{\partial T_i}{\partial y}(l) = -B_i A_i K \lambda\, e^{at} \cosh \lambda l \qquad (14)$$

A second relationship between ΔT_i and ΔQ_i can be found by differentiating equation (2) to obtain

$$\Delta Q_i = -\frac{(T_0 - T_i)}{{R_i}^2}\frac{dR_i}{dx}\Delta x - \frac{\Delta T_i(l)}{R_i} \qquad (15)$$

$$= -\frac{A_i\, K\, T_0}{(l + A_i\, K\, R_i)\, R_i}\frac{dR_i}{dx}\Delta x - \frac{\Delta T_i(l)}{R_i} \qquad (16)$$

from equation (3).

We now solve equation (5) for R_i, obtaining

$$R_i = \frac{l}{A_i K}\left(\frac{1}{f_i} - 1\right) \qquad (17)$$

and substitute into equation (16), from which

$$\frac{l}{A_i K}\left(\frac{1}{f_i} - 1\right)\Delta Q_i = \frac{T_0}{f_i}\frac{df_i}{dx}\Delta x - \Delta T_i(l). \qquad (18)$$

The function x, defined by equations (9*i*)–(9*iii*) is the difference between the unconstrained thermal expansions of the two rods, ($u_1 - u_2$) and hence

$$\Delta x = \Delta u_1 - \Delta u_2 = \alpha \int_0^l \{T_1(y) - T_2(y)\}\, dy$$
$$= \alpha(B_1 - B_2)\, e^{at}\, (\cosh \lambda l - 1)/\lambda \qquad (19)$$

from equation (12).

Two simultaneous equations can now be obtained by substituting

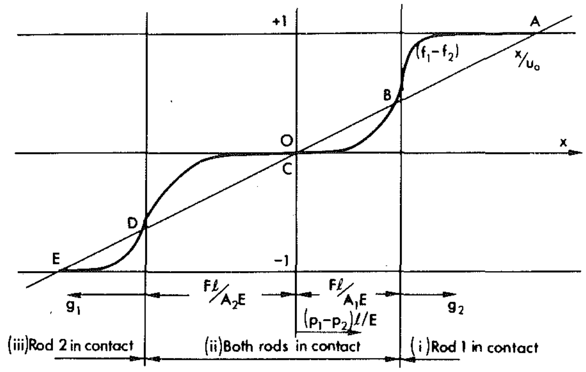

Fig. 3 Graphical solution of equation (10)

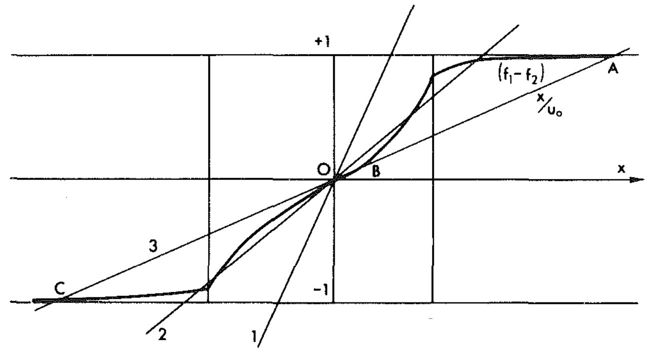

Fig. 4 Variation of contact resistance throughout the load range

for ΔQ_i, $\Delta T_i(l)$, Δx from equations (12), (14), (19) into equation (18), i.e.,

$$- B_1 \left(\frac{1}{f_1} - 1\right) z^2 \cosh z = \frac{2u_0 f_1'}{f_1} \times (B_1 - B_2)(\cosh z - 1) - B_1 z \sinh z \quad (20)$$

and

$$- B_2 \left(\frac{1}{f_2} - 1\right) z^2 \cosh z = \frac{2u_0 f_2'}{f_2} \times (B_1 - B_2)(\cosh z - 1) - B_2 z \sinh z \quad (21)$$

where

$$z = \lambda l \quad (22)$$

Finally, B_1, B_2 can be eliminated between these equations to give the characteristic equation for z (and hence a) which is

$$\frac{2u_0 f_1' (\cosh z - 1)}{(1 - f_1) z^2 \cosh z + f_1 z \sinh z} - \frac{2u_0 f_2' (\cosh z - 1)}{(1 - f_2) z^2 \cosh z + f_2 z \sinh z} = 1 \quad (23)$$

The perturbation (12) is unstable if, and only if, equation (23) has a root for which a and hence z^2 has a positive real part. This equation is investigated in the Appendix, where it is shown that

(*i*) There are no unstable complex roots and
(*ii*) Unstable real roots occur if, and only if,

$$(f_1' - f_2') > 1/u_0 \quad (24)$$

provided f_1, f_2 are monotonic functions of x.

Discussion

The relationship between the stability condition (24) and the steady-state solution equation (10) is illustrated graphically in Fig. 3.

The contact resistance is assumed to be continuous at the transition between contact and noncontact and hence $(f_1 - f_2)$ is a continuous function of x. This function is illustrated for the case in which the transition from perfect thermal contact to perfect insulation occurs over a small range of load or gap, in which case the curve passes nearly horizontally through the origin. This is probably a realistic physical assumption, but it is not necessary to the development of the argument.

Solutions of equation (10) are represented by the intersections ($ABCDE$) between the curve $(f_1 - f_2)$ and the straight line (x/u_0).

Furthermore, the stability criterion (24) shows that those solutions are stable for which the straight line crosses the curve from below when x is increasing. In view of the limits imposed on $(f_1 - f_2)$, it follows that

(*i*) There must be an odd number of solutions.
(*ii*) Stable and unstable solutions alternate with increasing x.
(*iii*) The outermost solutions are both stable.

The stable solutions ACE in Fig. 3 correspond to solutions in the contact regimes described as (*i*), (*ii*), (*iii*), respectively, given previously in this section. However, there are also two intermediate unstable solutions BD at which one rod carries most of the load while the other is either very lightly loaded or has a very small gap. We can define a limit to the contact resistance function R_i such that the change from thermal insulation to perfect thermal contact occurs over an infinitesimal range of gap or load. The function $(f_1 - f_2)$ will then correspond to states in which one rod is in perfect contact, carrying the total load F, while the other is in "imperfect contact" as defined by a similar limiting process in the treatment of problems with the reverse direction of heat flow [4]. The state of imperfect contact is defined by the conditions

$$p_i = 0; \quad g_i = 0; \quad 0 < f_i < 1 \quad (25)$$

These results support the hypothesis [5] that imperfect contact states are unstable when heat flows into the material of higher distortivity.

Fig. 3 has been drawn for a case in which all five intersections occur, but it is clear that if the straight line had a sufficiently large slope—corresponding to low values of u_0 and hence T_0—the only intersection would be C. In other words, when the temperature difference is small, the only permissible steady-state solution is that involving contact of both rods.

If the temperature is now increased, the slope of the straight line is reduced and, at some critical temperature, a pair of additional solutions such as AB—one stable, one unstable—will be introduced. The function $(f_1 - f_2)$ is not necessarily symmetrical about $x = 0$, since A_1 may differ from A_2. There will therefore generally be distinct temperature ranges with one, three and five solutions, respectively.

If the load F is increased, the two "steps" in $(f_1 - f_2)$ are displaced further from the origin. This has a similar effect to a reduction in temperature.

Fig. 4 illustrates a more general situation in which the contact resistances and hence $(f_1 - f_2)$ vary significantly over the entire load range, giving a nonzero slope near the origin (notice that $(f_1 - f_2)$ does not necessarily pass through the origin).

As temperature is increased, the same behavior is observed as in Fig. 3, with progression from one solution (line 1) to five solutions (line 2). However, with a further increase in temperature (line 3), the system passes into a new regime with three solutions. One of these (B) has both rods in contact but is *unstable*, whilst the other two (AC) involve contact at one rod only and are stable. In effect, Fig. 3 represents the limiting situation in which the temperature difference needed to initiate this new regime is very large.

Definition of an Energy Function

Following the same procedure as in reference [1], we can define an energy function

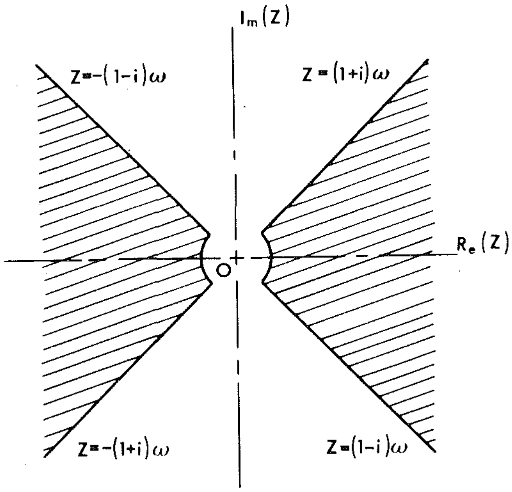

Fig. 5 Domain for unstable roots of equation (29)

$$U(x) = \frac{(A_1 + A_2)\,E}{l}\,[\int (f_1 - f_2)\,u_0 dx - \tfrac{1}{2}x^2] \quad (26)$$

The equation (10) for a steady-state solution can then be written

$$\partial U/\partial x = 0, \quad (27)$$

while the condition for instability (24) is

$$\partial^2 U/\partial x^2 < 0 \quad (28)$$

Thus $U(x)$ is stationary at all steady-state solutions, being a maximum if the solution is unstable and a minimum if it is stable.

The total mechanical energy for the system, calculated by Comninou and Dundurs [3] is not related to the function $U(x)$ and cannot be used to determine which solutions are stable. Indeed the aforementioned analysis shows that *all* the solutions which they consider—being those involving only perfect contact and separation—are stable. The only unstable solutions are those involving imperfect contact. These can be thought of as interposing higher energy barriers between the stable perfect contact/gap solutions.

References

1 Barber, J. R., Dundurs, J., and Comninou, M., "Stability Considerations in Thermoelastic Contact," ASME JOURNAL OF APPLIED MECHANICS, Vol. 47, 1980, pp. 871–874.

2 Comninou, M., and Dundurs, J., "On Lack of Uniqueness in Heat Conduction Through a Solid to Solid Interface," ASME *Journal of Heat Transfer*, Vol. 102, 1980, pp. 319–323.

3 Comninou, M., and Dundurs, J., "On the Possibility of History Dependence and Instabilities in Thermoelastic Contact," *Journal of Thermal Stresses*, Vol. 3, 1980, pp. 427–433.

4 Barber, J. R., "Contact Problems Involving a Cooled Punch," *Journal of Elasticity*, Vol. 8, 1978, pp. 409–423.

5 Comninou, M., and Dundurs, J., "On the Barber Boundary Conditions for Thermoelastic Contact," ASME JOURNAL OF APPLIED MECHANICS, Vol. 46, 1979, pp. 849–853.

APPENDIX

It is required to determine the conditions under which the equation

$$F(z) = \frac{2u_0 f_1'(\cosh z - 1)}{(1 - f_1)z^2 \cosh z + f_1 z \sinh z} - \frac{2u_0 f_2'(\cosh z - 1)}{(1 - f_2)z^2 \cosh z + f_2 z \sinh z} - 1 = 0 \quad (29)$$

has roots corresponding to values of z^2 with positive real part. In the z-plane, the corresponding zeros of $F(z)$ must lie in the two sectors shaded in Fig. 5 and bounded by the lines $z = \pm(1 + i)\,\omega$, the origin being excluded.

We first note that for the special case $f_1 = f_2 = 1; f_1' = f_2' = 0$, there are no such roots, since the only zeros of $F(z)$ correspond to

$$z \sinh z = 0 \quad (30)$$

and hence

$$z = 0 \pm in\pi \quad (31)$$

If we now allow f_1, f_2, f_1', f_2' to change continuously, the zeros will move continuously about the z-plane and will only be able to enter the unstable domain by crossing its boundaries.

(i) $\boldsymbol{z = (1 + i)\omega}$**.** The first term in $F(\omega + i\omega)$ can be written

$$F_1(\omega) = \frac{2u_0 f_1'(A + iB)}{(C + iD)} = \frac{2u_0 f_1'(A + iB)\,(C - iD)}{(C^2 + D^2)} \quad (32)$$

where

$$\begin{aligned} &A = ch \cdot c - 1; \quad B = sh \cdot s; \\ &C = -2\omega^2(1 - f_1)\, sh \cdot s + \omega f_1(sh \cdot c - ch \cdot s); \\ &D = 2\omega^2(1 - f_1)\, ch \cdot c + \omega f_1(sh \cdot c + ch \cdot s); \end{aligned} \quad (33)$$

and $s = \sin\omega$, $c = \cos\omega$, $sh = \sinh\omega$, $ch = \cosh\omega$

If f_1 is monotonic, $u_0 f_1' > 0$ and the imaginary part of $F_1(\omega)$ has the same sign as

$$\begin{aligned} CB - AD &= 2\omega^2(1 - f_1)\,(-sh^2s^2 - ch^2c^2 + ch \cdot c) + \omega f_1(sh^2cs \\ &\quad - s^2shch - c^2shch - ch^2cs + sh \cdot c + ch \cdot s) \\ &= \omega^2(1 - f_1)(s^2 - sh^2 - (c - ch)^2) \\ &\quad + \omega f_1\,(s - sh)(ch - c) < 0, (\omega > 0); \quad 0 \leqslant f_1 \leqslant 1 \end{aligned} \quad (34)$$

By similar argument, it can be shown that the second term in $F(\omega + i\omega)$ also has a negative imaginary part, since the function $f_2(x)$ must satisfy $f_2 < 0$.

It follows that $F(\omega + i\omega)$ has a negative (and hence nonzero) imaginary part for all $\omega > 0$ and no roots of equation (29) can therefore cross the line $z = \omega + i\omega$.

(ii) $\boldsymbol{z = \omega + i\delta}$**.** We next establish that no zeros can cross the line $z = \omega + i\delta$, where δ is small, and hence that all unstable roots are real.

Since $\delta \ll 1$, the corresponding forms of the coefficients in equation (32) are

$$\begin{aligned} &A = ch - 1; \quad B = \delta\, sh; \\ &C = \omega^2(1 - f_1)\, ch + \omega f_1 sh; \\ &D = \delta(1 - f_1)\,(2\omega ch + \omega^2 sh) + \delta f_1(sh + \omega ch) \end{aligned} \quad (35)$$

We therefore have

$$\begin{aligned} CB - AD &= (1 - f_1)(\omega^2 sh + 2\omega ch - 2\omega ch^2) \\ &\quad + \delta f_1\,(sh - \omega)(1 - ch) \\ &\quad < 0\ (\omega > 0) \end{aligned} \quad (36)$$

as can be demonstrated by expanding in powers of ω.

A similar argument applied to the second term in $F(\omega + i)$ enables us to conclude that no zeros can cross $z = \omega + i$ if $f_1' > 0$ and $f_2' < 0$.

The function $F(z)$ is even in z and hence its zeros must be symmetrically disposed with respect to the real and imaginary axes. It is not therefore necessary to prove corresponding results for the remaining boundaries.

(iii) $\boldsymbol{z = \delta + i0}$**.** The preceding arguments demonstrate that zeros can only enter the domain from the origin along the real axis and hence the stability boundary can be determined from the condition for a real root at $z = \delta + i0$, $\delta \ll 1$.

The function $F(z)$ can then be expanded in the form

$$F(\delta + i0) = u_0 f_1' - u_0 f_2' - 1 + 0(\delta^2) \quad (37)$$

and hence the stability boundary is defined by

$$u_0(f_1' - f_2') = 1 \quad (38)$$

The system is known to be stable for $f_1' = f_2' = 0$ and hence for instability we must have

$$u_0(f_1' - f_2') > 1. \quad (39)$$