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Short Communication

Similarity considerations in adhesive contact problems

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ABSTRACT

The classical 'JKR' solution for the force required to separate two elastic spheres is independent of the elastic moduli. We demonstrate that this remarkable simplification is a rigorous consequence of the quadratic character of the spherical profile and hence that it must also apply exactly to more general quadratic profiles such as the contact of two ellipsoids. The same feature is also responsible for the fact that Bradley's 'rigid sphere' solution depends only on the interface energy and not on the detailed character of the adhesive force law.

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1. Introduction

Contact problems involving adhesive (e.g. van der Waals) forces are of increasing importance, due to modern emphasis on nanoscale technology. The best known theoretical result of this form is the 'JKR' solution due to Johnson et al. [1] for the adhesive contact of a sphere and a plane. In particular, they show that the 'pull-off force' required to separate the bodies is given by

$$F_{\mathrm{JKR}} = \frac{3\pi R \Delta\gamma}{2}, \tag{1}$$

where R is the radius of the indenting sphere, the interface energy

$$\Delta\gamma = \int_0^\infty \mathcal{F}(z)\, dz, \tag{2}$$

and $\mathcal{F}(z)$ is the adhesive traction between two parallel surfaces separated by a distance z.

The JKR solution is strictly applicable only when the 'Tabor' parameter [2]

$$\mu = \left(\frac{R(\Delta\gamma)^2}{E^{*2} \epsilon^3} \right)^{1/3} \gg 1 \tag{3}$$

where E^* is the composite elastic modulus, and ϵ is a dimension characterizing the length over which the interatomic forces are significant. However, numerical treatments of the problem [3,4] show that the pull-off force varies rather modestly with μ. Indeed, if we go to the opposite extreme $\mu \to 0$ [for example by making the elastic modulus E^* very large], we recover the Bradley solution [5]

$$F_{\mathrm{Bradley}} = 2\pi R \Delta\gamma \tag{4}$$

for the adhesive force between a rigid sphere and plane, which exceeds the JKR prediction only by a factor of 4/3.

The most remarkable feature of these results is that both depend only on R and $\Delta\gamma$. In particular, the JKR force is independent of the elastic modulus, and the Bradley force depends on the force law $\mathcal{F}(z)$ only through its contribution to $\Delta\gamma$ in Eq. (2). These are clearly not general features of adhesive contact problems, since (for example) for a conical indenter the only dimensional parameters in the limit $\mu \to \infty$ are $\Delta\gamma$ and E^*, so from dimensional considerations the pull-off force must be a multiple of $(\Delta\gamma)^2/E^*$. Elementary calculations show that it is in fact $54(\Delta\gamma)^2/\pi\alpha^3 E^*$, where $\alpha \gg 1$ is the inclination of the cone face relative to the plane.

We shall show in this note that the reduced parametric dependence in both the JKR solution and the Bradley solution is a consequence of the quadratic shape of the contacting surfaces.

2. The generalized JKR problem

Suppose that the two elastic contacting bodies can be modelled as half spaces and that their profiles can be described by a composite initial gap function $g_0(\boldsymbol{x})$, where $\boldsymbol{x} = (x, y)$ defines a point on the interfacial plane. If the bodies were in fact rigid and were placed in contact, the interfacial plane would be the common tangent plane at the contact point and g_0 would be the distance between corresponding points on the two surfaces measured along the normal to this plane. We can then define a

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generalized JKR problem by the boundary conditions

$$\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{x}} = -\frac{\partial g_0(\boldsymbol{x})}{\partial \boldsymbol{x}}; \quad \boldsymbol{x} \in \mathcal{A} \tag{5}$$

$$\sigma(\boldsymbol{x}) = 0; \quad \boldsymbol{x} \notin \mathcal{A} \tag{6}$$

$$\iint_{\mathcal{A}} \sigma \, d\mathcal{A} = F \tag{7}$$

$$K_I = \lim_{\rho \to 0^+} \sigma(\boldsymbol{x}_B + \rho \boldsymbol{n})\sqrt{2\pi\rho} = \sqrt{2E^* \Delta\gamma}, \tag{8}$$

where $\mathcal{A}$ is the contact area, $\boldsymbol{x}_B$ is a point on the boundary of $\mathcal{A}$, $\boldsymbol{n}$ is the inward normal to the boundary at $\boldsymbol{x}_B$, u is the combined normal elastic displacement, σ is the normal contact traction, and F is the applied (tensile) force.

Notice that Eq. (8) defines a 'fracture mechanics' formulation of the condition at the boundary of the contact area, which is exactly equivalent to an energetic formulation of the problem in the 'JKR limit' $\mu \to \infty$.

2.1. A linear mapping

Suppose that the complete solution σ, u to the above problem is known for a case where the initial gap function can be expressed in polar coordinates (r, θ) as

$$g_0(r, \theta) = r^2 f(\theta), \tag{9}$$

where $f(\theta)$ is any continuous function of θ. We denote the boundary of the contact area $\mathcal{A}$ as $r = a(\theta)$. Notice that $a(\theta)$ and $f(\theta)$ will generally not have the same shape except in the special case where they are both constant (circular contact area). Even in the Hertzian elliptical case, the ellipticity of these two functions differs. Also, when adhesion is included, the shape of the contact area will generally change with force F, since the edge condition (8) and the contact condition (5) scale with different powers of the contact area dimensions.

We now define a geometrically similar displacement field $\hat{u}$ in the space defined by the linearly scaled coordinates

$$\hat{\boldsymbol{x}} = \lambda \boldsymbol{x}; \quad \hat{r} = \lambda r; \quad \hat{a} = \lambda a. \tag{10}$$

The new displacement field will satisfy conditions (5), (9) if and only if

$$\frac{\partial \hat{u}}{\partial \hat{r}} = -2\hat{r} f(\theta) = -2\lambda r f(\theta) = \lambda \frac{\partial u}{\partial r}, \tag{11}$$

or

$$\frac{\partial \hat{u}}{\partial \hat{\boldsymbol{x}}} = \lambda \frac{\partial u}{\partial \boldsymbol{x}}. \tag{12}$$

In other words, the strains must also scale with λ. Now suppose that the modulus scales with λ^n or

$$\hat{E} = \lambda^n E^*. \tag{13}$$

The corresponding stresses will then be defined by

$$\hat{\sigma} = \lambda^{n+1} \sigma \tag{14}$$

and the stress intensity factor in the new field is obtained as

$$\begin{aligned} \hat{K}_I &= \lim_{\hat{\rho} \to 0^+} \hat{\sigma}\sqrt{2\pi\hat{\rho}} = \lambda^{n+3/2} \lim_{\rho \to 0} \sigma\sqrt{2\pi\rho} \\ &= \lambda^{n+3/2} K_I. \end{aligned} \tag{15}$$

The scaled version of condition (8) is therefore

$$\lambda^{n+3/2} K_I = \sqrt{2\hat{E}\Delta\gamma} = \lambda^{n/2}\sqrt{2E^*\Delta\gamma} \tag{16}$$

and this will be satisfied if and only if $n = -3$. It then follows that $\hat{\sigma} = \lambda^{-2}\sigma$ and the total force

$$\begin{aligned} \hat{F} &= \int_0^{2\pi} \int_0^{\hat{a}(\theta)} \hat{\sigma}(\hat{r}, \theta)\hat{r} \, d\hat{r} \, d\theta \\ &= \int_0^{2\pi} \int_0^{a(\theta)} \sigma(r, \theta) r \, dr \, d\theta. \end{aligned} \tag{17}$$

In other words, the total force is unchanged by the scaling. We conclude that if the gap function has the assumed quadratic form and the modulus is changed by the factor $1/\lambda^3$, the contact area for a given applied force will remain the same shape, but its linear dimensions will increase in the proportion λ.

We also note that Eq. (12) implies that the transformed displacement $\hat{u} = \lambda^2 u$ and hence that the normal approach d of the bodies scales as

$$\hat{d} = \lambda^2 d. \tag{18}$$

It follows that the force–displacement relation for modulus $\hat{E}$ is identical with that the E^* except for a scaling through λ^2 *on the d-axis only*. Thus, the maximum tensile value of F (the pull-off force under force control) and the force at the point where $dd/dF = 0$ (the pull-off force under displacement control) are independent of the elastic modulus.

3. The Bradley solution

Bradley's solution for the adhesive force between a rigid sphere and a plane depends on the interface energy (2), but not on the details of the force law $\mathcal{F}(z)$. For the more general profile of Eq. (9), integrating the adhesive traction over the surface of the interfacial plane, we obtain

$$\begin{aligned} F &= \int_0^{2\pi} \int_0^{\infty} \mathcal{F}(g_0(r, \theta)) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{\infty} \mathcal{F}(r^2 f(\theta)) r \, dr \, d\theta. \end{aligned} \tag{19}$$

Writing

$$z = r^2 f(\theta); \quad dz = 2rf(\theta) \tag{20}$$

in the inner integral (for which θ and hence $f(\theta)$ is a constant), we have

$$\begin{aligned} F_{\text{Bradley}} &= \frac{1}{2}\int_0^{2\pi} \int_0^{\infty} \frac{\mathcal{F}(z) \, dz \, d\theta}{f(\theta)} \\ &= \frac{\Delta\gamma}{2}\int_0^{2\pi} \frac{d\theta}{f(\theta)}, \end{aligned} \tag{21}$$

using (2). Thus, the Bradley force remains independent of the detailed distribution $\mathcal{F}(z)$ for this case. However, it is clear from this derivation that the reduced parametric dependence depends on the factor r in dz is Eq. (20) and hence on the initial gap function being of quadratic form.

4. Discussion

The simplest non-spherical initial gap of the quadratic form (12) arises in the contact of an ellipsoid and a plane for which

$$g_0(r, \theta) = \frac{r^2(1 - e^2 \cos^2 \theta)}{2R}, \tag{22}$$

where R is the composite radius across the minor axis and e is the ellipticity of contours of the gap function g_0. Johnson and Greenwood [6] gave an approximate solution to the corresponding JKR problem in which the contact area was assumed to be elliptical and Eq. (8) was imposed only at the ends of the major and minor axes. Their results show that K_I then varies around the edge of the assumed contact area, the maximum deviation being 4% when the ellipticity of the contact area is 0.8. It follows that the correct contact area in this case cannot be an exact ellipse, though no exact solution has so far been proposed. However, the argument of Section 2 shows that the exact pull-off force must be independent of the elastic modulus and in fact must be of the form $CR\Delta\gamma$, where C is a dimensionless constant that depends only on e.

4.1. Plane problems

The above results apply specifically to three-dimensional problems, but one might hope to extract a corresponding result for plane (two-dimensional) problems by a limiting process in which the curvature in the y-direction (say) tends to zero, leading in the limit to the function $f(\theta) = \cos^2(\theta)$. The pull-off force must remain independent of the elastic modulus for all truly three-dimensional cases, but in the limit the problem becomes equivalent to the adhesive contact of a circular cylinder on a plane for which Chaudhury et al. [7] obtained a pull-off force of

$$F_{2D} = \frac{3}{4}[4\pi E^* R(\Delta\gamma)^2]^{1/3}, \tag{23}$$

which depends on the elastic modulus E^*.

The resolution of this paradox is that Chaudhury's result defines a force *per unit length* along the cylinder axis, whereas the three-dimensional analysis defines the total force which tends to infinity as the two-dimensional limit is approached. We could define an *average* force per unit length as $F_{2D} = F/2a$ where a is the semi-major axis of the quasi-elliptical contact area (it is not exactly elliptical as proved by Johnson and Greenwood [6]), but since $\hat{a} = \lambda a$, this would give

$$\hat{F}_{2D} = \frac{F}{2\hat{a}} = \frac{F}{2\lambda a} = F_{2D}\left(\frac{E^*}{\hat{E}}\right)^{1/3}, \tag{24}$$

agreeing with the modulus dependence in Chaudhury's result. The dependence on R and $\Delta\gamma$ then follows from dimensional considerations.

An analysis exactly similar to that in Section 2 can be applied to the two-dimensional problem, where the contact area comprises a line segment. In this case, the pull-off force is independent of modulus only for indentation by a (possibly unsymmetrical) wedge defined by $g_0(x) = -C_1 x$, $x < 0$ and $g_0(x) = C_2 x$, $x > 0$. Also, for this profile, the Bradley force depends only on the interface energy $\Delta\gamma$.

5. Conclusions

We have given a rigorous proof that the JKR pull-off force will be independent of the elastic modulus for any contact problem defined by the quadratic initial gap function of Eq. (9). For all other profiles, the JKR force will depend on the modulus and the Bradley force will depend on the form of intermolecular force law assumed.

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