STABILITY OF THERMOELASTIC CONTACT OF A LAYER AND A HALF-PLANE

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The conductive heat transfer between two different materials in contact can cause the system to be unstable due to the interaction between thermoelastic distortion and pressure-dependent interface resistance. This article investigates the stability of a system consisting of a layer and a half-plane using a perturbation method.

INTRODUCTION

It is well known that mathematical difficulties arise in the solution of steady-state thermoelastic contact problems if conventional idealized boundary conditions are applied [1]. Barber [2] and Duvaut [3] showed that difficulties of existence can be overcome by assuming a pressure-dependent thermal contact resistance at the interface, but multiple solutions are still possible with this boundary condition [2, 3].

Early studies of the stability of steady-state solution of such problems suggested that when the steady-state solution is unique, it is also stable, whereas when multiple solutions are obtained, they are alternately stable and unstable [4]. However, if the two contacting bodies are both deformable and have different thermal diffusivities, the stability behavior is more complex and examples can be found of unique steady-state solutions that are unstable [5].

More recently, the stability of the two-dimensional contact of two dissimilar half-planes has been investigated by examining the conditions under which a sinusoidal perturbation in contact pressure can grow exponentially with time [6, 7]. This technique was suggested by earlier work on related problems by Dow and Burton [8] and by Richmond and Huang [9]. The configuration has no characteristic length, so the stability behavior depends on material properties alone and can be classified depending on the relative magnitudes of the ratios of two thermal properties for the materials. Most of the material combinations considered exhibited one of two kinds of behavior. For the simpler of these, characterized as type 1, instability occurs only when the heat flows into the material of higher distortivity, which is also the condition required for nonuniqueness of the steady-state solution. However, for type 2 material combinations, instability can occur for either direction of heat flow if the

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NOMENCLATURE

b	exponential growth rate
h	layer thickness
H	dimensionless layer thickness
J	dimensionless thermomechanical parameter
k	thermal diffusivity
K	thermal conductivity
m	spatial frequency of sinusoidal perturbation
p_0	unperturbed contact pressure
Δp	perturbation in contact pressure
q_0	unperturbed heat flux
Δq	perturbation in heat flux
Q^*	dimensionless heat flux
r	ratio of material properties
R	pressure-dependent contact resistance
R^*	dimensionless thermal resistance
T	temperature
T^*	temperature drop
ΔT	perturbation in temperature drop
u	displacement
α	coefficient of thermal expansion
$\hat{\alpha}$	Dundurs' constant
δ	distortivity
μ	modulus of rigidity
ν	Poisson's ratio
ϕ	thermoelastic potential
σ	stress
ψ	isothermal potential

magnitude of the heat flux is sufficiently large, thus allowing the possibility of unique but unstable steady states.

The present article takes a first step toward investigating the effect of a finite geometry on the stability of thermoelastic contact by extending the results of Zhang and Barber [7] to the case where heat is conducted across an interface between an elastic layer and half-plane. A question of particular interest is whether the stability boundary is determined by the longest possible wavelength of perturbation—as in the case of two half-planes—or whether the "optimum" wavelength is related to the layer thickness.

STABILITY OF A LAYER AND A HALF-PLANE

The system consists of a layer, $0 \leq y \leq h$, and a half-plane, $y \leq 0$, which make contact at their common place $y = 0$ as shown in Fig. 1. The layer is pressed against

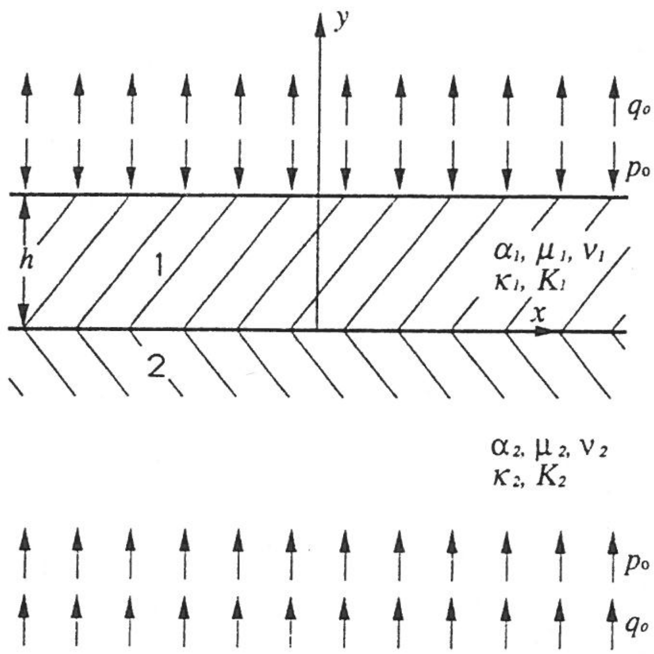

Fig. 1 A layer on a half-plane, pressed together and transmitting heat.

the half-plane by a uniform pressure p_0, and a uniform heat flux $q_y = q_0$ is imposed at the exposed surface $y = h$. The properties of the layer and half-plane are identified by the subscripts 1 and 2, respectively.

Temperature Perturbation

As in [6], we investigate the conditions under which the sinusoidal perturbation in temperature,

$$T = f(y)e^{bt} \cos mx \tag{1}$$

can grow exponentially with time. The temperature perturbation must satisfy the transient heat conduction equation

$$\nabla^2 T = \frac{1}{k}\frac{\partial T}{\partial t} \tag{2}$$

where k_i $(i = 1, 2)$ is the thermal diffusivity of the material.

Substituting (1) into (2) and solving for $f(y)$, we find that the perturbation in temperature can be written as

$$T_1 = e^{bt}(A_{11}e^{-a_1 y} + A_{12}e^{a_1 y}) \cos mx \qquad \text{for the layer} \tag{3}$$

$$T_2 = e^{bt} A_2 e^{a_2 y} \cos mx \qquad \text{for the half-plane} \tag{4}$$

where A_{ij} and A_i are arbitrary constants and

$$a_i^2 = m^2 + \frac{b}{k_i} \qquad a_1, a_2 > 0 \tag{5}$$

In the event that a_i is complex, it is defined to have positive real part. In Eq. (4), we take only the positive exponential term because the perturbation in temperature must decay as $y \to -\infty$.

Thermoelastic Stresses and Displacements

A particular solution of the thermoelastic problem [10] corresponding to the above temperature field can be obtained in terms of a thermoelastic potential ϕ, where

$$2\mu \mathbf{u}_i = \nabla \phi_i \tag{6}$$

$$\nabla^2 \phi_i = \frac{2\mu_i \alpha_i (1 + \nu_i) T_i}{(1 - \nu_i)} \tag{7}$$

and α_i, μ_i, and ν_i are, respectively, the coefficient of thermal expansion, modulus of rigidity, and Poission's ratio of material i.

It can be verified by substitution that potential functions which satisfy (3), (4), and (7) are given by

$$\phi_1 = \left[\frac{J_1 K_1}{b} (A_{11} e^{-a_1 y} + A_{12} e^{a_1 y}) + C_{11} e^{-my} + C_{12} e^{my} \right] e^{bt} \cos mx \tag{8}$$

$$\phi_2 = \left[\frac{J_2 K_2}{b} A_2 e^{a_2 y} + C_2 e^{my} \right] e^{bt} \cos mx \tag{9}$$

where the dimensionless thermomechanical parameter J_i is defined by [11]

$$J_i = \frac{2\alpha_i \mu_i (1 + \nu_i) k_i}{(1 - \nu_i) K_i}$$

and K_i is the thermal conductivity. In Eq. (9), we take only the positive exponential term because the thermoelastic stresses must decay away from the contact plane.

The corresponding tractions and normal displacement in the layer are

$$\begin{aligned} 2\mu_1 u_{y1} &= \frac{\partial \phi_1}{\partial y} \\ &= \left[\frac{J_1 K_1}{b} a_1 (-A_{11} e^{-a_1 y} + A_{12} e^{a_1 y}) - mC_{11} e^{-my} + mC_{12} e^{my} \right] e^{bt} \cos mx \end{aligned} \tag{10}$$

$$\sigma_{xy1} = \frac{\partial^2 \phi_1}{\partial x\, \partial y}$$

$$= -m\left[\frac{J_1 K_1}{b} a_1(-A_{11}e^{-a_1 y} + A_{12}e^{a_1 y}) - mC_{11}e^{-my} + mC_{12}e^{my}\right]e^{bt}\sin mx \quad (11)$$

$$\sigma_{yy1} = -\frac{\partial^2 \phi_1}{\partial x^2}$$

$$= m^2\left[\frac{J_1 K_1}{b}(A_{11}e^{-a_1 y} + A_{12}e^{a_1 y}) + C_{11}e^{-my} + C_{12}e^{my}\right]e^{bt}\cos mx \quad (12)$$

Similarly, for the half-plane we obtain

$$2\mu_2 u_{y2} = \frac{\partial \phi_2}{\partial y} = \left(\frac{J_2 K_2}{b} a_2 A_{12}e^{a_2 y} + C_2 m e^{my}\right)e^{bt}\cos mx \quad (13)$$

$$\sigma_{xy2} = \frac{\partial^2 \phi_2}{\partial x\, \partial y} = -m\left(\frac{J_2 K_2}{b} a_2 A_2 e^{a_2 y} + C_2 m e^{my}\right)e^{bt}\sin mx \quad (14)$$

$$\sigma_{yy2} = -\frac{\partial^2 \phi_2}{\partial x^2} = m^2\left(\frac{J_2 K_2}{b} A_2 e^{a_2 y} + C_2 e^{my}\right)e^{bt}\cos mx \quad (15)$$

The Contact Problem

The heat flux and traction at the upper surface of the layer are prescribed to be uniform and hence the *perturbations* in these quantities are zero; i.e.,

$$\sigma_{xy1} = \sigma_{yy1} = 0 \qquad q_1 = 0 \quad \text{at} \quad y = h \quad (16)$$

We also require that there be frictionless contact and that the heat flux be continuous at the interface $y = 0$ and hence

$$\sigma_{xy1} = \sigma_{xy2} = 0 \qquad \sigma_{yy1} = \sigma_{yy2} \qquad u_{y1} = u_{y2} \qquad q_1 = q_2 \qquad \text{at } y = 0 \quad (17)$$

The solution of Eqs. (10)–(15) is not sufficiently general to satisfy these mechanical boundary conditions. We must therefore superpose an additional isothermal solution, a suitable form being provided by solution B of Green and Zerna [12] with the harmonic potentials

$$\psi_1 = (B_{11}e^{-my} + B_{12}e^{my})e^{bt}\cos mx \quad (18)$$

$$\psi_2 = B_2 e^{my} e^{bt}\cos mx \quad (19)$$

The additional traction and displacement components are then

$$2\mu_1 u_{y1} = y\frac{\partial\psi_1}{\partial y} - (3-4\nu_1)\psi_1$$

$$= [ym(-B_{11}e^{-my} + B_{12}e^{my}) - (3-4\nu_1)(B_{11}e^{-my} + B_{12}e^{my})]e^{bt}\cos mx \tag{20}$$

$$\sigma_{xy1} = y\frac{\partial^2\psi_1}{\partial x\,\partial y} - (1-2\nu_1)\frac{\partial\psi_1}{\partial x}$$

$$= [-ym^2(-B_{11}e^{-my} + B_{12}e^{my}) + (1-2\nu_1)m(B_{11}e^{-my} + B_{12}e^{my})]e^{bt}\sin mx \tag{21}$$

$$\sigma_{yy1} = y\frac{\partial^2\psi_1}{\partial y^2} - 2(1-\nu_1)\frac{\partial\psi_1}{\partial y}$$

$$= [m^2y(B_{11}e^{-my} + B_{12}e^{my}) - 2(1-\nu_1)m(-B_{11}e^{-my} + B_{12}e^{my})]e^{bt}\cos mx \tag{22}$$

for the layer ($0 \le y \le h$) and

$$2\mu_2 u_{y2} = y\frac{\partial\psi_2}{\partial y} - (3-4\nu_2)\psi_2 = B_2[my - (3-4\nu_2)]e^{my}e^{bt}\cos mx \tag{23}$$

$$\sigma_{xy2} = y\frac{\partial^2\psi_2}{\partial x\,\partial y} - (1-2\nu_2)\frac{\partial\psi_2}{\partial x} = B_2[-m^2y + (1-2\nu_2)m]e^{my}e^{bt}\sin mx \tag{24}$$

$$\sigma_{yy2} = y\frac{\partial^2\psi_2}{\partial y^2} - 2(1-\nu_2)\frac{\partial\psi_2}{\partial y} = B_2[m^2 - 2(1-\nu_2)m]e^{my}e^{bt}\cos mx \tag{25}$$

for the half-plane ($y \le 0$).

Boundary Conditions

Superposing the solutions of Eqs. (10)–(15) and imposing the boundary conditions (16) and (17), we get a system of nine equations:

$$-(-A_{11} + A_{12})a_1K_1 = -A \tag{26}$$

$$-A_2a_2K_2 = -A \tag{27}$$

$$A_{11}e^{a_1h} - A_{12}e^{a_1h} = 0 \tag{28}$$

$$-m\left[\frac{J_2K_2}{b}a_1(-A_{11} + A_{12}) - mC_{11} + mC_{12}\right] + (1-2\nu_1)m(B_{11} + B_{12}) = 0 \tag{29}$$

$$-m\left(\frac{J_2K_2}{b}a_2A_1 + C_2m\right) + (1-2\nu_2)mB_2 = 0 \tag{30}$$

$$-m\left[\frac{J_1K_1}{b}a_1(-A_{11}e^{-a_1h}+A_{12}e^{a_1h})-mC_{11}e^{-mh}+mC_{12}e^{mh}\right]$$

$$+[-m^2h(-B_{11}e^{-mh}+B_{12}e^{mh})+(1-2\nu_2)m(B_{11}e^{-mh}+B_{12}e^{mh})]=0 \quad (31)$$

$$m^2\left[\frac{J_1K_1}{b}(A_{11}e^{-a_1h}+A_{12}e^{a_1h})+C_{11}e^{-mh}+C_{12}e^{mh}\right]$$

$$+\{[m^2h+2(1-\nu_1)m]B_{11}e^{-mh}+[m^2h-2(1-\nu_1)m]B_{12}e^{mh}\}=0 \quad (32)$$

$$m^2\left[\frac{J_1K_1}{b}(A_{11}+A_{12})+C_{11}+C_{12}\right]-2(1-\nu_1)m(-B_{11}+B_{12})$$

$$-m^2\left(\frac{J_2K_2}{b}A_2+c_2\right)-2(1-\nu_2)mB_2=0 \quad (33)$$

$$\frac{1}{2\mu_1}\left[\frac{J_1K_1}{b}a_1(-A_{11}+A_{12})-mC_{11}+mC_{12}-(3-4\nu_1)(B_{11}+B_{12})\right]$$

$$-\frac{1}{2\mu_2}\left[J_2\frac{A_1a_2}{b}+C_2m-(3-4\nu_2)B_2\right]=0 \quad (34)$$

These equations can be solved for A_2, B_2, C_2, A_{11}, A_{12}, B_{11}, B_{12}, C_{11}, and C_{12} in terms of A. The perturbations in contact pressure, temperature difference, and heat flux at the interface can then be determined by substituting for these constants in the expressions

$$\Delta p=-\sigma_{yy}=\left[-m^2\left(\frac{J_2K_2}{b}A_2+C_2\right)+2m(1-\nu_2)B_2\right]e^{bt}\cos mx \quad (35)$$

$$\Delta T=T_2-T_1=(A_2-A_{11}-A_{12})e^{bt}\cos mx \quad (36)$$

$$\Delta q=-Ae^{bt}\cos mx \quad (37)$$

Perturbation of the Thermal Resistance Relation

To complete the solution, we linearize the equation defining heat conduction across the thermal contact resistance, R, for small perturbations about the steady state.

The definition of the pressure-dependent contact resistance, R, implies that

$$q_y=\frac{T^*}{R(p)} \quad (38)$$

where T^* is the temperature drop across the interface. Hence for small perturbations about the steady state, we have

$$R_0\,\Delta q + q_0\,\Delta R = \Delta T \tag{39}$$

Finally, noting that

$$\Delta R = R'\,\Delta p \tag{40}$$

we can substitute for ΔT, Δq, and Δp from Eqs. (35)–(37) to obtain the characteristic equation for the exponential growth rate b.

It is convenient to cast this equation in dimensionless form by defining dimensionless layer thickness, thermal resistence, and heat flux, respectively, through the relations

$$H = mh \qquad R^* = R_0 m K_1 \qquad Q^* = -4\alpha_1(1 + \nu_1)q_0 R' M \tag{41}$$

where

$$\frac{1}{2M} = \frac{(1 - \nu_1)}{\mu_1} + \frac{(1 - \nu_2)}{\mu_2} \tag{42}$$

and the ratios of material properties

$$r_1 = \frac{k_2}{k_1} \qquad r_2 = \frac{\delta_2}{\delta_1} \qquad r_3 = \frac{K_2}{K_1}$$

where the distortivity $\delta_i = \alpha_i(1 + \nu_i)/K_i$.

With this notation, the characteristic equation can be written as

$$R^* + D_1(H, z)Q^* + D_2(H, z) = 0 \tag{43}$$

where

$$D_1 = -\left[\frac{1}{C_1(C_1 + 1)} - \frac{r_2}{C_2(C_2 + 1)}\right]$$

$$+ \frac{r_2}{(C_2^2 - 1)\{1 - [(1 - \hat{\alpha})(2H^2 + 2H + 1) - 4H]e^{-2H} - \hat{\alpha}e^{-4H}\}}$$

$$\cdot\left\{\frac{1}{r_1 r_2 C_1}\left(\frac{1}{\tanh C_1 H} - 1\right) + (4He^{-2H} - e^{-4H})\right.$$

$$\cdot\left(\frac{1+\hat{\alpha}}{1-\hat{\alpha}}\frac{1}{C_2}+\frac{1}{r_1r_2C_1}\frac{1}{\tanh C_1H}\right)-\frac{2e^{-H}}{r_1r_2C_1\sinh C_1H}[(H+1)+(H-1)e^{-2H}]$$

$$-\left[\frac{-2}{r_1r_2}+(1+\hat{\alpha})(2H^2+2H+1)\right.$$

$$\left.+((1-\hat{\alpha})(2H^2+2H+1)-4H)\left(\frac{1}{r_1r_2}-\frac{1+\hat{\alpha}}{1-\hat{\alpha}}\frac{1}{C_2}-\frac{1}{r_1r_2C_1}\right)\right]e^{-2H}$$

$$\left.-\left[(1+\hat{\alpha})\left(\frac{1}{r_1r_2}-1\right)-\hat{\alpha}\left(\frac{1+\hat{\alpha}}{1-\hat{\alpha}}\frac{1}{C_2}+\frac{1}{r_1r_2C_1}\right)\right]e^{-4H}\right\}$$

$$D_2=\left(\frac{1}{C_1}+\frac{1}{r_3C_2}\right)+\left(\frac{1}{\tanh C_1H}-1\right)\frac{1}{C_1} \qquad (44)$$

$$C_1=\frac{a_1}{m}=\left(1+\frac{b}{m^2k_1}\right)^{1/2}=(1+z)^{1/2} \qquad (45)$$

$$C_2=\frac{a_2}{m}=\left(1+\frac{b}{m^2k_2}\right)^{1/2}=\left(1+\frac{z}{r_1}\right)^{1/2} \qquad (46)$$

and $\hat{\alpha}$ is Dundurs' constant [13], defined by†

$$\hat{\alpha}=\frac{\mu_2(1-\nu_1)-\mu_1(1-\nu_2)}{\mu_2(1-\nu_1)+\mu_1(1-\nu_2)} \qquad (47)$$

We can recover the characteristic equation for the problem of two half-planes by allowing H to approach infinity, in which case the second terms of D_1 and D_2 become zero.

STABILITY CRITERION

The system of Fig. 1 will be unstable if and only if the characteristic equation (43) has at least one solution for the dimensionless exponential growth rate $z(=b/m^2k_1)$ that is positive or complex with positive real part. The system is clearly stable for

†It is interesting to note that $\hat{\alpha}$ appears only in terms that tend to zero as the layer thickness $H \to \infty$ and hence has no influence on the stability criterion for the thermoelastic contact of two half-planes.

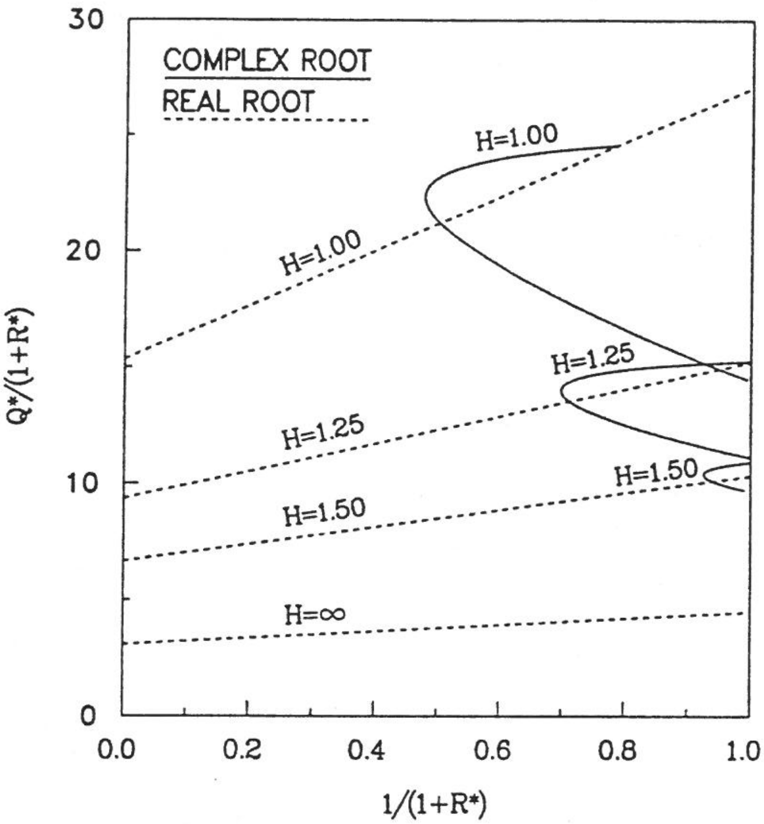

Fig. 2 Stability boundaries as a function of R^* (aluminum alloy layer on a copper half-plane).

$Q^* = 0$, and hence if Q^* is increased monotonically, instability will be indicated when the first root of Eq. (43) passes into the right half-plane, either through the origin or by crossing the imaginary axis.

In the former case, the stability criterion is determined by setting $z = 0$, in which case Eq. (43) defines a linear relation between R^* and Q^*, that is,

$$Q^* = -\frac{R^* + D_2(H, 0)}{D_1(H, 0)} \tag{48}$$

As $z \to 0$, C_1 and $C_2 \to 1$ and both the numerator and denominator of D_1 approach zero. However, we can use L'Hôpital's rule to recover the limiting form of $D_1(H, 0)$, which is

$$D_1(H, 0) = \frac{r_2}{2} + \left[-\left(\frac{1}{\tanh H} + \frac{H}{\sinh^2 H}\right)(1 + 4He^{-2H} - e^{-4H}) \right.$$

$$+ 4e^{-2H}\left(\frac{H \cosh H + \sinh H}{\sinh H}\right)^2 - r_2(1 + \hat{\alpha})(2H^2 + 2H + 1)e^{-2H}$$

$$\left. + r_2(1 + \hat{\alpha})e^{-4H} \right] \Big/ 2\{1 - [(1 - \hat{\alpha})(2H^2 + 2H + 1) - 4H]e^{-2H} - \hat{\alpha}e^{-4H}\} \tag{49}$$

We also note that

$$D_2(H, 0) = \frac{1}{r_3} + \frac{1}{\tanh H} \tag{50}$$

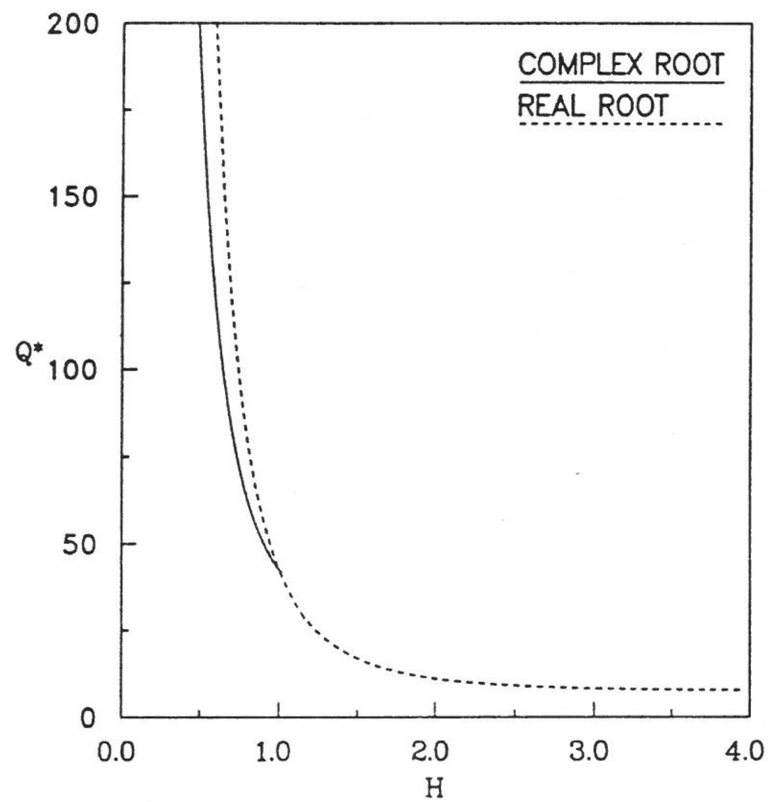

Fig. 3 Stability boundaries as a function of H when $R^* = 1.0$ (aluminum alloy layer on a copper half-plane).

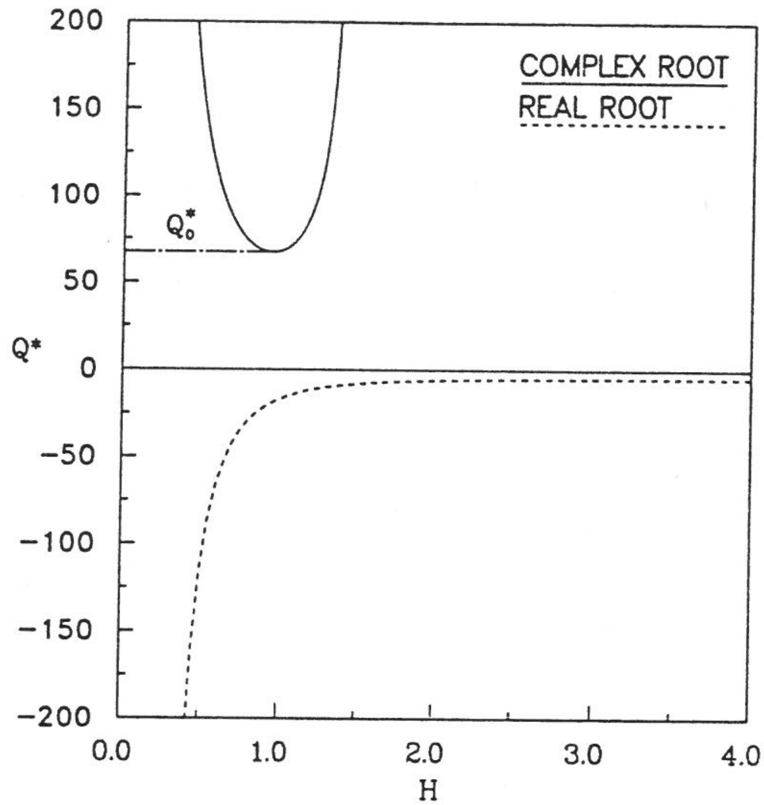

Fig. 4 Stability boundaries as a function of H when $R^* = 1.0$ (copper layer on an aluminum alloy half-plane).

When the roots reach the unstable half-plane by crossing the imaginary axis, the characteristic equation (43) becomes complex and cannot be solved explicitly. However, separating real and imaginary parts and noting that Q^*, and R^* must be real, we obtain the two real equations

$$R^* + \mathrm{Re}(D_1)Q^* + \mathrm{Re}(D_2) = 0 \tag{51}$$

$$\mathrm{Im}(D_1)Q^* + \mathrm{Im}(D_2) = 0 \tag{52}$$

from which

$$Q^* = -\frac{\mathrm{Im}(D_2)}{\mathrm{Im}(D_1)} \tag{53}$$

$$R^* = -\mathrm{Re}(D_1)Q^* - \mathrm{Re}(D_2) \tag{54}$$

These equations can be solved parametrically, with $z = jw$ as parameter, to determine the relation between Q^* and R^* at the stability boundary.

For both real and complex cases, we find that $D_1 \to O(H)$ and $D_2 \to O(H^{-1})$ as $H \to 0$. It therefore follows from Eqs. (48) and (53) that a very large $[O(H^{-2})]$ heat flux Q^* is required to cause instability when the layer thickness H is small.

RESULTS

In discussing specific cases, it is convenient to make use of the classification of material properties introduced by Zhang and Barber [7] for the limiting case of the contact of two half-planes.

Type 1 Material Combinations

If $r_1 > 1$ and $0 < r_2 < 1/r_1$, the contact of two half-planes is unstable only for heat flow into the more distortive material, the critical value of Q^* being determined by the condition for a root occuring at the origin.

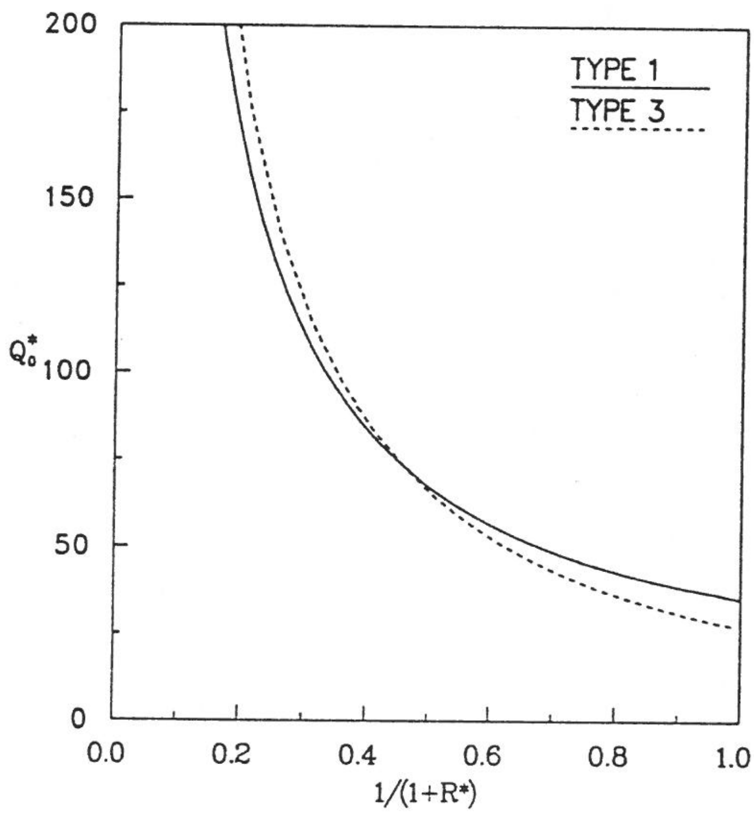

Fig. 5 Minimum heat flow Q_0^* into less distortive material for instability in case of type 1 and type 3 combinations.

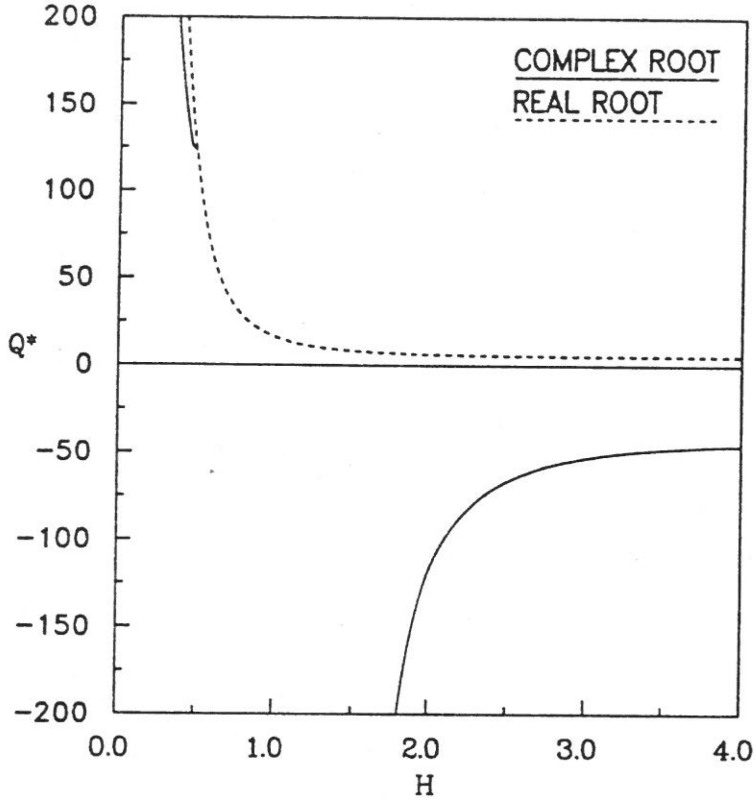

Fig. 6 Stability boundaries as a function of H when $R^* = 1.0$ (stainless steel layer on an aluminum alloy half-plane).

Figure 2 shows the corresponding stability chart for finite values of H, for the case of an aluminum alloy layer and a copper half-plane. For large values of H, stability continues to be determined by the linear real root criterion (48), but for thinner layers and smaller values of R^*, a complex instability can be obtained at lower values of Q^*. More detailed investigation of Eq. (43) shows that once a root has crossed into the right half-plane, it will remain there with increasing Q^*, so the stability criterion is determined by the lower of the curves in Fig. 2 for the appropriate H. The critical Q^* is shown as a function of H at constant R^* (= 1.0) in Fig. 3, where we note that, as predicted, $Q^* \to \infty$ as $H \to 0$.

The problem of two half-planes is unchanged if the two materials are interchanged, provided the direction of heat flow is also reversed. This is not the case when one of the bodies is replaced by a layer of finite thickness. Figure 4 shows the effect of H on the minimum Q^* for instability for a copper layer on an aluminum alloy half-plane, with $R^* = 1.0$. Note that with the labeling convention of the present article, the type 1 material combination is then determined by $r_1 < 1$ and $0 < 1/r_2 < r_1$.

At large H, we again find that instability occurs only for heat flow into the more distortive material (this time the half-plane, making $Q^* < 0$), but at lower values of H, a new feature is that instability can also occur for the opposite direction of heat flow, being determined by a complex root. Corresponding plots for different values of R^* show that the minimum Q^* ($= Q_0^*$) required to precipitate this kind of instability always occurs for a layer thickness of approximately $H = 1$, but the corresponding critical Q^* decreases with decreasing R^*, as shown in Fig. 5.

The dimensionless layer thickness is defined by $H = mh$ and hence $H = 1$ corresponds to $h = 1/m = l/2\pi$, where l is the wavelength of the perturbation. Thus, another interpretation of Fig. 4 is that instability for heat flow into the layer is most likely to be associated with wavelengths of the order of $2\pi h$.

Type 2 Material Combinations

Type 2 material combinations are characterized by the conditions $1/r_1 < r_2 < 1$, if $r_1 > 1$ or $r_1 < 1/r_2 < 1$ if $r_1 < 1$ and they exhibit instability for both directions of heat flow, even for the case of two half-planes. This behavior extends to the case of the layer, as shown in Fig. 6, for a stainless steel layer on an aluminum alloy

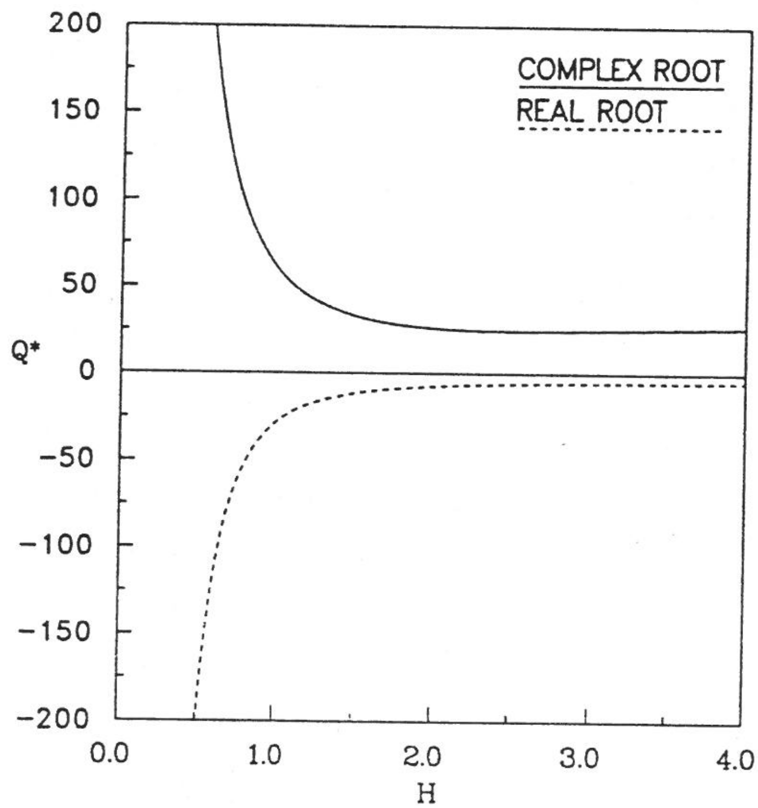

Fig. 7 Stability boundaries as a function of H when $R^* = 1.0$ (aluminum alloy layer on a stainless steel half-plane).

half-plane. Note that, as in Fig. 3, the boundary for the heat flow into the more distortive material (the stainless steel) is determined by a complex root at small values of H. However, this effect is not observed when the materials of the layer and the half-plane are interchanged (see Fig. 7).

Type 3 Material Combinations

Type 3 material combinations are defined by the conditions $1 < r_2 < r_1$ or $1 > r_2 > r_1$, and permit instability only for heat flow into the more distortive material for the case of two half-planes, the stability criterion being determined by a complex root. Figure 8 shows the stability boundary as a function of H for a cast-iron layer on a brass half-plane and $R^* = 1.0$. As in Fig. 3, a region of instability for the opposite direction of heat flow is obtained, with a minimum near $H = 1$. The value of Q^* associated with this minimum is shown in Fig. 5 as a function of R^*. When the materials of the layer and the half-plane are interchanged (Fig. 9), this effect is no longer observed and instability occurs only for heat flow into the more distortive material.

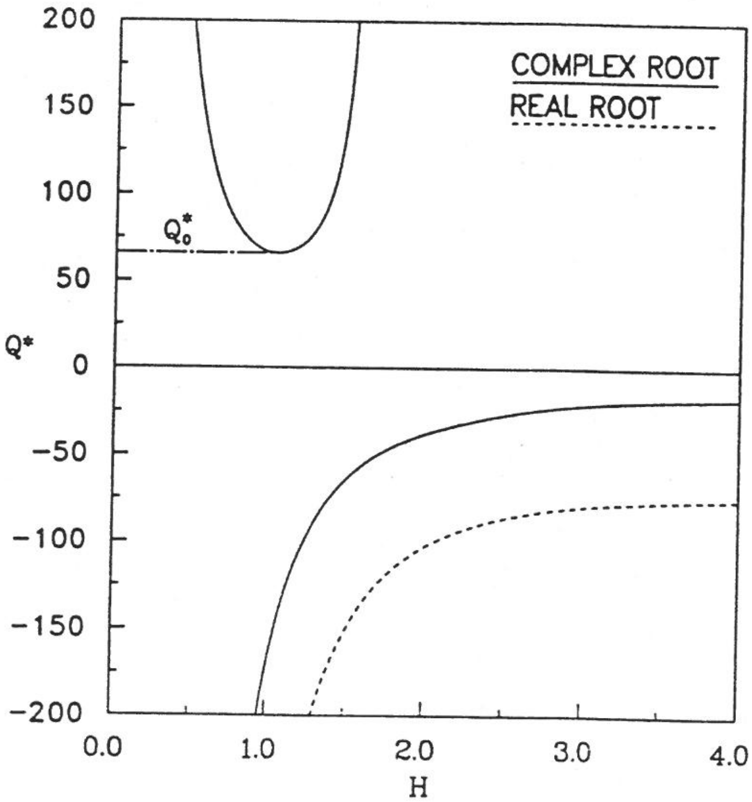

Fig. 8 Stability boundaries as a function of H when $R^* = 1.0$ (cast-iron layer on a brass half-plane).

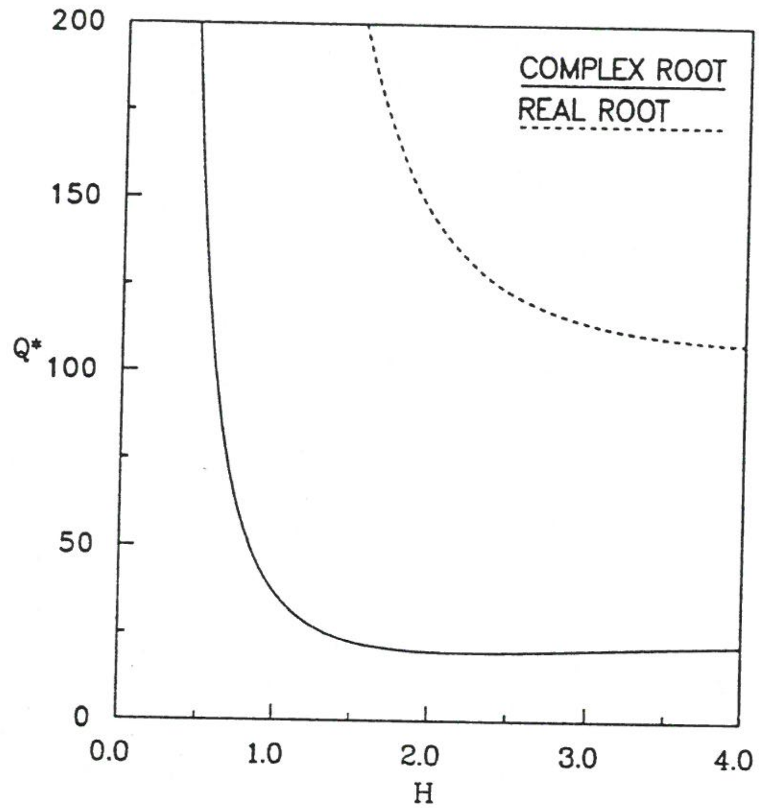

Fig. 9 Stability boundaries as a function of H when $R^* = 1.0$ (brass layer on a cast-iron half-plane).

CONCLUSIONS

The above results extend the analysis of Zhang and Barber [7] to include the effect of a finite body dimension—the thickness of a layer—on the stability of thermoelastic contact.

As in earlier studies, instability is found to be most often associated with heat flow into the more distortive material except when the ratios of the material properties are characterized as type 2. However, with the layer geometry, additional examples of instability for the heat flow into the *less* distortive material occur when this is the material of the layer. These conditions are associated with complex roots and typically involve perturbations whose wavelengths are of the order $2\pi h$, where h is the layer thickness.

For very thin layers ($H \ll 1$), the system is generally stable, in the sense that a very large heat flux is needed to cause instability.

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