GREEN'S FUNCTIONS FOR TRANSIENT THERMOELASTIC CONTACT PROBLEMS FOR THE HALF-PLANE*

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Summary

The most straightforward method of setting up thermoelastic contact problems is to find the surface displacement of an equivalent traction-free body and then to solve an isothermal contact problem for this distorted body. In this paper, Green's functions are given which permit the solution of the former problem to be expressed as a double integral of the surface heat input or temperature in time and space. The functions are chosen and defined with a view to efficient numerical computations, since the solutions of transient thermoelastic contact problems tend to be lengthy.

1. Introduction

Of the various methods for solving elastic contact problems for the half-plane, the most direct is to write down the governing integral equations in terms of appropriate Green's functions. This method has been extensively used by Comninou and coworkers [1 - 3], who have listed the Green's functions for steady state thermoelastic problems [4, 5].

The same approach is relatively even more advantageous in the corresponding transient problems, since the temperature is then not a harmonic function. The only problems of this type which have so far been solved are associated with the phenomenon of thermoelastic instability [6 - 9], but there are also many cases of technological importance in "static" (*i.e.* non-sliding) contact, *e.g.* the thermal "shock" stresses developed when a hot body is placed in contact with a cold one.

Some of the Green's functions needed for an integral equation formulation were given in ref. 10. In this paper, these results are extended to

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include tangential displacements needed for the solution of adhesive and frictional contact problems. The corresponding results are also given for heat sources distributed over a finite region of time and space which permit the integral equations to be solved numerically by a suitable discretization of the contact area. Some attention is given to the question of the efficient numerical computation of these functions.

2. The instantaneous line source

The fundamental solution required corresponds to the release at time $t = 0$ of a quantity of heat Q per unit length along the line $x = y = 0$ on the surface of the half-plane $y > 0$, which is initially at zero temperature (Fig. 1).

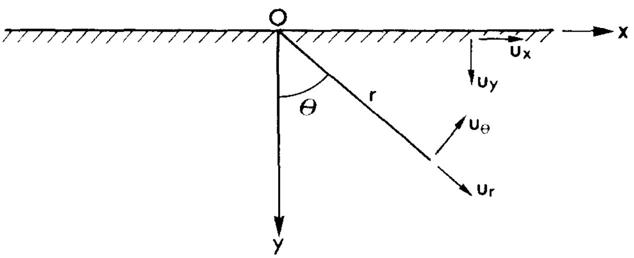

Fig. 1. Configuration and coordinate system for the half-plane.

If there is subsequently no heat flow across the surface plane, the temperature distribution will be

$$T = \frac{Q \exp(-R^2)}{2\pi k\rho ct} \tag{1}$$

(ref. 11, Section 10.3) where k, ρ and c are the thermal diffusivity, the density and the specific heat of the half-plane respectively and

$$R^2 = \frac{r^2}{4kt} = \frac{x^2 + y^2}{4kt} \tag{2}$$

A particular solution to the thermal stress problem can be found [10] by applying eqn. (1) to the infinite body and using the solution (ref. 12, Section 151) for a solid cylinder with an axisymmetric temperature distribution.

In polar coordinates r, θ, the stresses and displacements are

$$\sigma_{rr} = -\frac{\alpha E}{(1-\nu)r^2} \int_0^r Tr\, \mathrm{d}r = -\frac{\alpha QE}{4\pi\rho c(1-\nu)kt} \frac{1-\exp(-R^2)}{R^2} \tag{3}$$

$$\sigma_{\theta\theta} = \frac{\alpha E}{1-\nu}\left(\frac{1}{r^2}\int_0^r Tr\, \mathrm{d}r - T\right) = \frac{\alpha QE}{4\pi\rho c(1-\nu)kt}\left\{\frac{1-\exp(-R^2)}{R^2} - 2\exp(-R^2)\right\} \tag{4}$$

$$u_r = \frac{\alpha(1+\nu)}{(1-\nu)r}\int_0^r Tr\,\mathrm{d}r = \frac{\alpha Q(1+\nu)}{2\pi\rho c(1-\nu)(kt)^{1/2}}\,\frac{1-\exp(-R^2)}{R} \tag{5}$$

where α, E and ν are the coefficient of thermal expansion, Young's modulus and Poisson's ratio respectively for the half-plane. The shear stress $\sigma_{r\theta}$ and the circumferential displacement u_θ are zero, from considerations of symmetry.

The solution of eqns. (3) - (5) corresponds to the case where the surface $y = 0$ is constrained to remain plane by the traction distribution $\sigma_{\theta\theta}$. The solution for a traction-free surface is obtained by superposing an equal and opposite traction, using the Flamant solution for a point force (ref. 12, Section 36). This corrective solution imposes a normal displacement at the surface of

$$u_y = -\frac{2(1-\nu^2)}{\pi E}\int_{-\infty}^{+\infty}\sigma_{\theta\theta}(s)\log|r-s|\,\mathrm{d}s \tag{6}$$

neglecting the rigid body term: thus

$$u_y = -\frac{\alpha Q(1+\nu)}{\pi^2\rho c(kt)^{1/2}}\int_{-\infty}^{+\infty}\left\{\frac{1-\exp(-S^2)}{S^2} - 2\exp(-S^2)\right\}\log|R-S|\,\mathrm{d}S \tag{7}$$

$$= -\frac{\alpha Q(1+\nu)}{\pi\rho c(kt)^{1/2}}F_1(R) \tag{8}$$

where

$$F_1(R) = \frac{2}{\pi^{1/2}}\,\frac{\exp(-R^2)}{R}\int_0^R \exp(S^2)\,\mathrm{d}S \tag{9}$$

The function $F_1(R)$ is related to the complex error function discussed by Miller and Gordon [13]. Series and asymptotic forms are given in Appendix A.

The corresponding tangential displacement is obtained in the same way and must be added to the radial displacement from the symmetric solution (eqn. (5)) to give

$$u_x = \frac{\alpha Q(1+\nu)}{\pi\rho c(kt)^{1/2}}\,\frac{1-\exp(-X^2)}{X} \qquad y = 0 \tag{10}$$

where

$$X = \frac{x}{(4kt)^{1/2}} \tag{11}$$

3. Distributed sources

The instantaneous source solution is appropriate for the analytic formulation of transient thermoelastic contact problems and will lead to a double integral equation in time and space. However, the kernel of this equation will be one of the functions in eqns. (1), (8) and (10), all of which are singular at the origin for $t = 0$. These singularities prevent a direct numerical attack on the integral equation, but the difficulty can be overcome, and the convergence of the solution improved, by developing the corresponding results for a source uniformly distributed over a discrete region of time and space.

The distributed solutions are functions of two variables, the size of the heated region and its distance from the origin, but some simplification can be achieved by expressing all results in terms of sums and differences between several cases of the solution for a source extending from the origin to a distant point. This solution is a function of a single variable except for a possible multiplying power of x or t.

3.1. The instantaneous distributed source

Suppose that a quantity of heat q per unit area is released at time $t = 0$ over the region $0 < x < s$ (Fig. 2). The displacement u_y at the origin is obtained by integrating eqn. (8) and is

$$u_y = -\frac{\alpha q(1+\nu)}{\pi\rho c(kt)^{1/2}} \int_0^S F_1(X)\,\mathrm{d}x \tag{12}$$

$$= -\frac{\alpha q(1+\nu)}{\pi\rho c} F_2(S) \tag{13}$$

where

$$F_2(S) = 2\int_0^S F_1(X)\,\mathrm{d}X \tag{14}$$

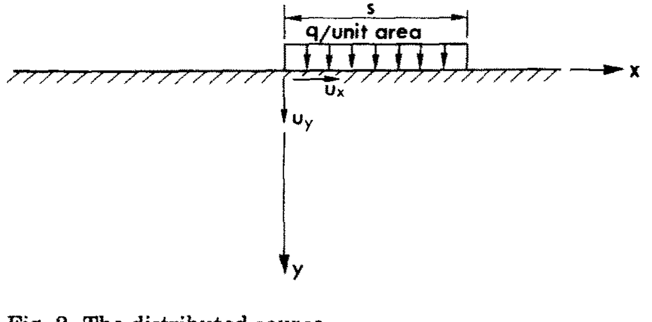

Fig. 2. The distributed source.

In particular, we note that $F_2(S) \to \pi$ as $S \to \infty$. Series and asymptotic forms are given in Appendix A.

The corresponding results for tangential displacement and temperature can be obtained in the same way and are

$$u_x = -\frac{\alpha q(1+\nu)}{\pi \rho c}\{-\mathrm{Ei}(-S^2) + C + \ln(S^2)\} \tag{15}$$

$$T = \frac{q}{2\rho c(\pi k t)^{1/2}}\,\mathrm{erf}(S) \tag{16}$$

where $C = 0.577216\ldots$ is Euler's constant, $\mathrm{Ei}(x)$ is the exponential integral and $\mathrm{erf}(x)$ is the probability integral (ref. 14, Sections 8:21 and 8:25).

3.2. The continuous point source

Consider the point source distributed in time; a quantity of heat q per unit time is released at the origin, starting at time $t = 0$. The displacement at a distance x from the source at time t is then

$$u_y = -\frac{\alpha q(1+\nu)}{\pi \rho c}\int_0^t \frac{F_1[x/\{4k(t-\tau)\}^{1/2}]\,\mathrm{d}\tau}{\{k(t-\tau)\}^{1/2}} \tag{17}$$

From eqn. (8)

$$u_y = -\frac{\alpha q(1+\nu)(kt)^{1/2}}{\pi k \rho c}F_3(X) \tag{18}$$

where the integral

$$F_3(X) = 2X\int_X^\infty \frac{F_1(S)\,\mathrm{d}S}{S^2} \tag{19}$$

can be expressed in terms of F_1 and F_2 as

$$F_3(X) = XF_2(X) + F_1(X) - \pi|X| + \frac{2}{\pi^{1/2}} \tag{20}$$

and is related to the function $L(X)$ of ref. 10 through

$$\pi F_3(X) = -L(X) \tag{21}$$

The corresponding results for tangential displacement and temperature are

$$u_x = \frac{\alpha q(1+\nu)(kt)^{1/2}}{\pi k \rho c}\left\{\frac{1-\exp(-X^2)}{X} + X\mathrm{Ei}(-X^2)\right\} \tag{22}$$

$$T = -\frac{q\mathrm{Ei}(-X^2)}{2\pi k \rho c} \tag{23}$$

(*cf.* ref. 11, Section 10.4).

3.3. The continuous distributed source

Finally, consider the case of the source distributed in time and space. The geometry is as shown in Fig. 2 with heat released at q per unit area per unit time, starting at $t = 0$. The results are conveniently obtained by integrating the results of Section 3.2 in space. At the origin (*i.e.* the left-hand end of the source) at time t we have

$$u_y = -\frac{\alpha q(1+\nu)t}{\pi\rho c} F_4(X) \tag{24}$$

$$u_x = -\frac{\alpha q(1+\nu)t}{\pi\rho c} F_5(X) \tag{25}$$

$$T = \frac{q(kt)^{1/2}}{\pi k\rho c} F_6(X) \tag{26}$$

where

$$F_4(X) = 2\int_0^X F_3(S)\,\mathrm{d}S \tag{27}$$

$$F_5(X) = (X^2 - 1)\mathrm{Ei}(-X^2) - \{1 - \exp(-X^2)\} + \ln(X^2) + C \tag{28}$$

and

$$F_6(X) = \pi^{1/2}\,\mathrm{erf}(X) - X\mathrm{Ei}(-X^2) \tag{29}$$

The function F_4 can be expressed in terms of F_1 and F_2 as

$$F_4(X) = (1 + X^2)F_2(X) + XF_1(X) - \pi X|X| + \frac{2X}{\pi^{1/2}} \tag{30}$$

4. Numerical implementation

A thermoelastic contact problem can be discretized in time and space by dividing the contact area into n strips of width δa, as shown in Fig. 3, and proceeding in equal increments δt of time.

If the heat input rate to the ith strip in the jth time increment is q_{ij}, the value of u_y at the midpoint of the kth strip in the lth time increment can be written

$$u_y = \sum_{i=1}^{n}\sum_{j=1}^{l} c_{ijkl}q_{ij} \tag{31}$$

where the coefficient c_{ijkl} is essentially the displacement at $x = 0$, $t = 0$, due to a unit heat input over the single strip $x_1 < x < x_2$ extending in time over $-t_2 < t < -t_1$, where

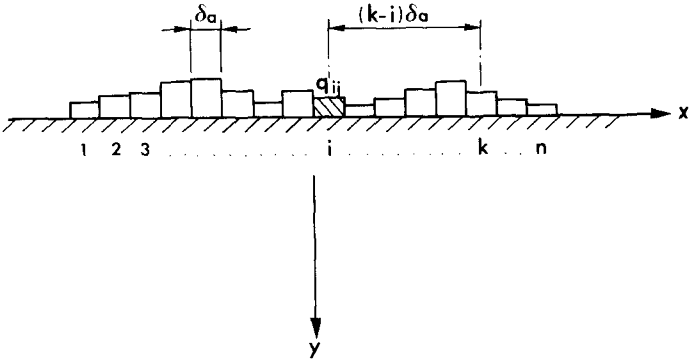

Fig. 3. Discretization of the contact area.

$$x_1 = \left(i - k - \frac{1}{2}\right)\delta a \qquad x_2 = \left(i - k + \frac{1}{2}\right)\delta a$$
$$t_1 = \left(l - j - \frac{1}{2}\right)\delta t \qquad t_2 = \left(l - j + \frac{1}{2}\right)\delta t \tag{32}$$

This solution can be written down by superposition from eqn. (24) as

$$c_{ijkl} = -\frac{\alpha(1+\nu)}{\pi\rho c}\left(t_2\left[F_4\left\{\frac{x_2}{(4kt_2)^{1/2}}\right\} - F_4\left\{\frac{x_1}{(4kt_2)^{1/2}}\right\}\right] - \right.$$
$$\left. - t_1\left[F_4\left\{\frac{x_2}{(4kt_1)^{1/2}}\right\} - F_4\left\{\frac{x_1}{(4kt_1)^{1/2}}\right\}\right]\right) \qquad j \neq l \tag{33}$$
$$= \frac{\alpha(1+\nu)}{\pi\rho c}\left(\frac{1}{2}\delta t\left[F_4\left\{\frac{x_2}{(2k\,\delta t)^{1/2}}\right\} - F_4\left\{\frac{x_1}{(2k\,\delta t)^{1/2}}\right\}\right]\right) \qquad j = l \tag{34}$$

Equation (34) can be subsumed under the general form (33) by adopting the convention that the function F_4 is zero when its argument is imaginary (*i.e.* when $t_1 < 0$).

The corresponding expression for u_x is of the same form as eqns. (31) - (33) with F_5 replacing F_4, whilst for temperature we have

$$T = \frac{1}{\pi\rho c k^{1/2}} \sum_{i=1}^{n} \sum_{j=1}^{l} \left(t_2^{1/2}\left[F_6\left\{\frac{x_2}{(4kt_2)^{1/2}}\right\} - F_6\left\{\frac{x_1}{(4kt_1)^{1/2}}\right\}\right] - \right.$$
$$\left. - t_1^{1/2}\left[F_6\left\{\frac{x_2}{(4kt_2)^{1/2}}\right\} - F_6\left\{\frac{x_1}{(4kt_1)^{1/2}}\right\}\right]\right) \tag{35}$$

If $\delta a \ll x_1$ or $\delta t \ll t_1$, the numerical evaluation of eqns. (33) - (35) may be inexact, since small differences are being found between large quantities. In the limit $\delta a \to 0$, eqn. (33) can be defined in terms of the derivative of the function F_4 and the numerical difficulties are essentially those encountered in numerical differentiation.

This problem was investigated for the function F_4 by comparing $F_4(X + \delta X/2) - F_4(X - \delta X/2)$ and $2F_3(X)\,\delta X$, with a double-precision calculation of the former, for various values of X and δX. The direct calculation using F_4 was more accurate in the range $\delta X > 0.02X$ but for smaller ratios the function F_3 should be used. The accuracy given by F_3 increases when X is small. For example, at $X = 0.2$, an accuracy of 1 in 10^3 is achieved on using the point source solution even for adjacent elements ($\delta X/X = 1$). This level of accuracy is achieved by F_3 for all X if $\delta X/X < 1/16$.

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Appendix A

Series and asymptotic forms for the functions $F_i(X)$

The functions $F_i(X)$ defined by eqns. (9), (14), (19) and (27) all permit a convergent series expansion for small X and an asymptotic series representation at large X.

The first nine terms of the asymptotic series give an accuracy of six significant figures in the range $X > 3$ at the lower end of which the convergent series requires 31 terms to achieve the same accuracy. Only nine terms are required for the same accuracy at $X = 1$. For a more efficient evaluation, an approximation based on Chebyshev polynomials could be developed in the range $1 \leqslant X \leqslant 3$ (see ref. A1).

The convergent series are

$$F_1(X) = \frac{2}{\pi^{1/2}} \sum_{i=0}^{\infty} \frac{(-2X^2)^i}{(2i+1)!!} \tag{A1}$$

$$F_2(X) = \frac{4}{\pi^{1/2}} \sum_{i=0}^{\infty} \frac{(-2X^2)^i X}{(2i+1)!!\,(2i+1)} \tag{A2}$$

$$F_3(X) = -\pi|X| - \frac{4}{\pi^{1/2}} \sum_{i=0}^{\infty} \frac{(-2X^2)^i}{(2i-1)(2i+1)!!} \tag{A3}$$

$$F_4(X) = -\pi X|X| - \frac{8}{\pi^{1/2}} X \sum_{i=0}^{\infty} \frac{(-2X^2)^i}{(2i+1)!!(4i^2-1)} \tag{A4}$$

where

$$(2i+1)!! \equiv 1 \times 3 \times 5 \times \ldots \times (2i+1)$$

The asymptotic series for large values of X are

$$F_1(X) \approx \frac{2}{\pi^{1/2}} \sum_{i=0}^{N} \frac{(2i-1)!!}{(2X^2)^{i+1}} \tag{A5}$$

$$F_2(X) \approx \pi - \frac{2}{\pi^{1/2}X} \sum_{i=0}^{N} \frac{(2i-1)!!}{(2i+1)(2X^2)^i} \tag{A6}$$

$$F_3(X) \approx \frac{4}{\pi^{1/2}} \sum_{i=0}^{N} \frac{(2i-1)!!}{(2i+3)(2X^2)^{i+1}} \tag{A7}$$

$$F_4(X) \approx \pi - \frac{4}{\pi^{1/2}X} \sum_{i=0}^{N} \frac{(2i-1)!!}{(2i+1)(2i+3)(2X^2)^i} \tag{A8}$$

where $(-1)!! = +1$.

Corresponding results for the functions $\mathrm{Ei}(-X^2)$ and $\mathrm{erf}(X)$ are given in ref. A2, but the approximating functions given in ref. A1 are numerically more efficient.

References for Appendix A

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