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# THE TRANSIENT THERMOELASTIC CONTACT OF A SPHERE SLIDING ON A PLANE*

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## Summary

A solution based on a Hertzian approximation to the contact pressure distribution is obtained for the transient thermoelastic contact of a sphere sliding on a rigid non-conducting plane. With this simplification an integral equation formulation is obtained. In a non-dimensional form the only independent parameter is the starting condition and numerical results are presented for various values.

An analytical solution is obtained for short values of time and shows that the contact radius falls linearly at a rate dependent only on the steady state contact radius and the thermal diffusivity of the sphere.

## 1. Introduction

The occurrence of non-uniform contact pressures between nominally conforming sliding solids on a scale larger than that of the individual asperity was first demonstrated by the now classical experiments of Parker and Marshall [1]. They used low temperature radiation pyrometry to measure the surface temperature of tread braked railway wheels and observed that at any instant the frictional heat input appeared to be concentrated into narrow bands parallel to the direction of sliding. Subsequent experimental investigations have demonstrated that a similar load concentration occurs in a wide range of unlubricated sliding systems with particularly serious consequences for the design of brakes (see *e.g.* ref. 2).

An explanation [3, 4] based on the potentially unstable interaction between frictional heating, thermal distortion and contact pressure was proposed in 1967, and the phenomenon is therefore now generally known as "thermoelastic instability". Considerable attention has been paid to the

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theoretical analysis of this process during the last 12 years, particularly by the research group of Professor R. A. Burton of Northwestern University. Amongst other interesting results, they have shown [5] that for a two-dimensional geometry a small sinusoidal disturbance in contact pressure between two conforming sliding solids will only grow if the sliding speed exceeds a critical value which depends on the wavelength. A similar criterion was developed by Burton *et al.* [6, 7] for the sliding speed necessary to sustain a steady state configuration in which the load is concentrated into a patch of a given width, the rest of the surface being out of contact. However, no solution has been given for the transient process whereby an initially small pressure disturbance develops into a condition of patch-like contact. This is the problem which will be considered in this paper.

Apart from establishing the link between the initiation of a disturbance and its steady state condition, the transient process is of practical interest in that braking systems operate for a limited period of time and hence it is possible that in many applications the severe thermal conditions described by the steady state solution are not achieved.

## 2. Statement of the problem

The problem to be considered is that of sliding contact between two elastic half-spaces, one of which is a thermal insulator. It is assumed that the coupling between tangential and normal traction at the interface can be neglected and that the coefficient of friction is constant throughout the contact area. The non-conducting solid can also be assumed to be rigid without loss of generality.

Experimental evidence suggests that the thermoelastic contact area is generally circular except where the nominal contact area imposes a constraint as in seals, *e.g.* tread braked railway wheels exhibit circular thermally affected zones on the surface [8]. We shall therefore consider in detail the case in which the temperature and stress fields in the conducting half-space are symmetrical about an axis. However, the corresponding two-dimensional problem could be treated by a similar method and would not be expected to give qualitatively different results.

With an axisymmetric geometry, a more natural interpretation of Burton's concept of a critical speed is to state that there is a certain size of steady state contact area associated with any given speed but if this is larger than the nominal contact area of the sliding solids there is no scope for thermoelastic instability to occur. The radius of this steady state contact area [9] is

$$a_0 = \frac{\pi K(1-\nu)}{1.566\alpha\mu GV(1+\nu)} \tag{1}$$

where $\alpha$, $K$, $G$ and $\nu$ are the coefficient of thermal expansion, thermal conductivity, modulus of rigidity and Poisson's ratio respectively for the deformable solid, $\mu$ is the coefficient of friction and $V$ is the sliding speed.

The conducting solid is assumed to be slightly rounded so as to give an initially Hertzian contact with a radius larger than $a_0$.

## 3. Method of solution

The thermoelastic contact problem at any instant can be treated by superposition of (1) the solution for the thermal stresses and deformations induced in the solid by the given temperature field, the contact surfaces being assumed to be stress free, and (2) the solution for the isothermal contact of a solid of this deformed shape with a rigid plane.

The normal surface displacement $u(r, t)$ of the half-space at radius $r$ and time $t$ depends upon the distribution of heat input $q(r, t_1)$ at all previous times ($t_1 < t$) and can be expressed as a convolution integral of this function by using the properties of the solution for a point source of heat [10]. The solution of the isothermal contact problem then defines the contact pressure distribution $p(r, t)$ in terms of $u(r, t)$, and since

$$q(r, t) = \mu V p(r, t) \tag{2}$$

the functions $p$ and $u$ can in principle be eliminated from this system of equations to give an integral equation in $q(r, t)$. However, the resulting equation contains a double integral with a difficult kernel and its solution is complicated by the necessity for introducing continuity conditions to determine the instantaneous contact radius $a(t)$.

We shall therefore develop an approximate solution based on the representation of the thermally deformed solid by a quadratic surface, *i.e.* the contact pressure at any instant will be approximated by the nearest Hertzian distribution. This permits the contact radius to be obtained explicitly. Thus if the traction-free profile of the solid at time $t$ (solution of problem (1)) is approximated by the expression

$$u(r, t) = C_0 + C_1 r^2 \tag{3}$$

the contact radius $a(t)$ will be given by

$$a^3 = \frac{3P(1-\nu)}{16GC_1} \tag{4}$$

and the contact pressure distribution by

$$p(r, t) = \frac{8GC_1(a^2 - r^2)^{1/2}}{\pi(1-\nu)} \tag{5}$$

(see for example ref. 11). The coefficients $C_0$ and $C_1$ are functions of time, the former being a rigid body displacement which has no effect on the contact problem. The frictional heat input per unit time due to this pressure distribution is

$$q(r, t) = \frac{8\mu VGC_1(a^2 - r^2)^{1/2}}{\pi(1-\nu)} \tag{6}$$

from eqn. (2).

## 4. The thermoelastic deformation

To complete the solution we need to find the thermoelastic normal surface displacement due to the frictional heat input of eqn. (6).

Results have already been published [10] for the displacement due to various distributed transient heat sources on the stress-free surface of a conducting half-space. In particular a quantity of heat $Q$ released at the origin at time $t = 0$ produces a normal surface displacement

$$u(r, t) = -\frac{Q\alpha(1+\nu)}{4\pi kt}\,\Phi\left(\frac{3}{2}, 2, -\frac{r^2}{4kt}\right) \qquad t > 0 \tag{7}$$

(eqn. (16) of ref. 10) where $k$ is the thermal diffusivity of the material and $\Phi$ is a degenerate hypergeometric function (9.21 of ref. 12).

The displacement due to the axisymmetric distributed heat source of eqn. (6) will therefore be

$$u(r, t) = -\frac{\alpha(1+\nu)}{4\pi k}\int_0^t\int_0^a\int_0^{2\pi}\Phi\left\{\frac{3}{2}, 2, -\frac{r^2+s^2-2rs\cos\theta}{4k(t-t_1)}\right\}\times$$

$$\times\frac{q(s, t_1)s\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}t_1}{t-t_1}$$

$$= -\frac{1}{0.783\pi a_0}\int_0^t\int_0^a\int_0^{2\pi}\Phi\left\{\frac{3}{2}, 2, -\frac{(r^2+s^2-2rs\cos\theta)}{4k(t-t_1)}\right\}\times$$

$$\times\frac{C_1(t_1)(a^2-s^2)^{1/2}s\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}t_1}{t-t_1} \tag{8}$$

from eqns. (1) and (6).

If the hypergeometric function is expanded in series form the two inner integrals can be performed to give

$$u(r, t) = \frac{8k}{0.783a_0}\int_0^t\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(2i+1)!!}{(2i)!!}\,\frac{(-1)^{i+1}}{(i+1)!}\,({}^iC_j)^2\times$$

$$\times\frac{(2i-2j)!!}{(2i-2j+3)!!}\left(\frac{r}{a}\right)^{2j}A^{2i+2}aC_1(t_1)\mathrm{d}t_1 \tag{9}$$

where

$$A^2 = \frac{a(t_1)^2}{4k(t-t_1)} \tag{10}$$

$$(2i+1)!! = 1 \times 3 \times 5 \times \ldots \times (2i+1)$$

and ${}^{i}C_{j}$ is the binomial coefficient.

To approximate this expression to the quadratic form of eqn. (3) we invert the order of summation with respect to $i$, $j$ and truncate the series in $j$ to two terms obtaining

$$u(r, t) = \frac{8k}{0.783a_0} \int_0^t \sum_{j=0}^{1} \sum_{i=j}^{\infty} \frac{(2i+1)!!(-1)^{i+1}({}^{i}C_{j})^2}{(2i)!!(i+1)!} \times \frac{(2i-2j)!!}{(2i-2j+3)!!}\left(\frac{r}{a}\right)^{2j} A^{2i+2} a C_1(t_1)\mathrm{d}t_1 \tag{11}$$

and hence

$$C_1(t) = \frac{8k}{0.783a_0} \int_0^t \sum_{i=1}^{\infty} \frac{i(-A^2)^{i+1} C_1(t_1)\mathrm{d}t_1}{2(i+1)!\, a(t_1)} + C_1(0) \tag{12}$$

$$= \frac{a_0}{0.783} \int_0^T \frac{\{1-(1+A^2)\exp(-A^2)\} C_1(T_1)\mathrm{d}T_1}{a(T_1)} + C_1(0) \tag{13}$$

where $C_1(0)$ is the initial curvature of the half-space and

$$T_1 = 4kt_1/a_0^2 \tag{14}$$

is a non-dimensional measure of time in terms of which

$$A^2 = \frac{\{a(T_1)/a_0\}^2}{T - T_1} \tag{15}$$

Finally we recast this integral equation in terms of the physically more interesting variable, the contact radius $a$, using eqn. (4) to obtain

$$\frac{0.783}{b(T)^3} - \frac{0.783}{b(0)^3} = \int_0^T \frac{\{1-(1+A^2)\exp(-A^2)\}\mathrm{d}T_1}{b(T_1)^4} \tag{16}$$

where

$$b = a/a_0 \tag{17}$$

## 5. Numerical solution

A numerical solution of eqn. (16) can be obtained by progressing in finite increments of time $T$ during each of which $a(T)$ can be regarded as constant. Equation (16) is thereby transformed into a recurrence relation between the $(n+1)$th value of $b$ and the first $n$ values. Repeated solution of this relation leads to the development of a series of values of $b(T)$ describing the progressive reduction in the contact radius with time.

The process can only start if an initial value $b(0)$ which corresponds to the existence of a slight curvature of the surface of the unheated and unstressed half-space is given for $b$ in the first time increment. Without such an initial perturbation, *i.e.* if $C_1(0) = 0$ and $b(0)$ is infinite, eqn. (16) has the trivial analytic solution $C_1(T) = 0$ for all $T$ which corresponds to a uniform distribution of contact pressure over the half-space.

## 6. The steady state solution

At large values of $T$ the contact radius approaches asymptotically a limit $a_1$ which can be found by integration of eqn. (16). In the steady state $a_1$ will be constant and hence from eqn. (15)

$$\mathrm{d}T_1 = -\frac{2a_1^2\,\mathrm{d}A}{a_0^2 A^3} \tag{18}$$

Substituting this result into eqn. (16) and allowing $T$ to tend to infinity gives

$$0.783 = \frac{2a_1}{a_0}\int_0^{\infty}\{1-(1+A^2)\exp(-A^2)\}\frac{\mathrm{d}A}{A^3}$$

$$= a_1/a_0 \tag{19}$$

Therefore

$$a_1 = 0.783a_0 \tag{20}$$

The value $a_0$ given by eqn. (1) is the contact radius obtained from an exact treatment of the steady state thermoelastic contact problem and hence the 22% difference between $a_1$ and $a_0$ gives some measure of the approximation involved in representing the contact pressure distribution as Hertzian.

## 7. The short time solution

If the non-dimensional time $T$ is small $A$ will be large (see eqn. (15)), and hence the expression $\{1-(1+A^2)\exp(-A^2)\}$ in the integrand of eqn. (16) will approximate to unity, giving

$$\frac{0.783}{b(T)^3} - \frac{0.783}{b(0)^3} = \int_0^T \frac{\mathrm{d}T_1}{b(T_1)^4} \tag{21}$$

This equation is easily shown to have the solution

$$b(T) = b(0) - 0.4257T \tag{22}$$

or in dimensional terms

$$a(t) = a(0) - 0.4257(4kt/a_0) \tag{23}$$

In other words the initial reduction of contact radius is linear with time, the rate of reduction being determined by the time scale associated with the appropriate steady state radius. Since this short time solution depends upon $A$ being large, its range of applicability is greater if $b(0)$ is large.

The function $\{1 - (1 + A^2)\exp(-A^2)\}$ differs from unity only in the sixth figure for $A > 4$ and the approximate solution (22) will therefore be accurate to at least this level provided that

$$\frac{b(0) - 0.4257T}{(T - T_1)^{1/2}} < 4 \qquad \text{for } 0 < T_1 < T \tag{24}$$

This condition can be shown to be satisfied as long as

$$\begin{aligned} &b(T) > 9.4 && b(0) > 18.8 \\ &b(T) > \{1 - 26.6\,b(0)\}\,b(0) && b(0) < 18.8 \end{aligned} \tag{25}$$

We can use this result to investigate the solution when the initial curvature is infinitesimal ($b(0) \to \infty$). It is then necessary to measure $T$ from a finite reference point. If this is chosen such that $b(0) > 9.4$, eqn. (16) becomes

$$\frac{0.783}{b(T)^3} = \int_{-\infty}^{T} \frac{\{1 - (1 + A^2)\exp(-A^2)\}\mathrm{d}T_1}{b(T_1)^4} \tag{26}$$

$$= \int_{-\infty}^{0} \frac{\{1 - (1 + A^2)\exp(-A^2)\}\mathrm{d}T_1}{\{b(0) - 0.4257T_1\}^4} + \int_{0}^{T} \frac{\{1 - (1 + A^2)\exp(-A^2)\}\mathrm{d}T_1}{b(T_1)^4} \tag{27}$$

If the terms in $A$ in the first integral can be replaced by unity this equation will be found to reduce to the same form as eqn. (16). In other words the response of the system beyond the linear region will be independent of the magnitude of the initial perturbation. The condition on $A$ requires that

$$T < \tfrac{1}{4}\, b(0)^2 \tag{28}$$

for this to hold, after which differences in the sixth figure may start to occur.

## 8. Results

The non-dimensional form of the controlling integral equation (16) shows that the only independent parameter is the initial value $b(0)$. In other words all combinations of physical constants and operating conditions are described by a set of responses for different $b(0)$. Furthermore, the above analysis shows that beyond a certain point the effect of increasing $b(0)$ is merely to extend the initial linear portion of the response except at large values of $T$.

These results are confirmed by numerical results, some of which are presented in Fig. 1 where the linear short time solution is shown by the chain line in each case. It will be observed that significant deviation from the linear short time solution only occurs when $b$ has fallen below about 3.

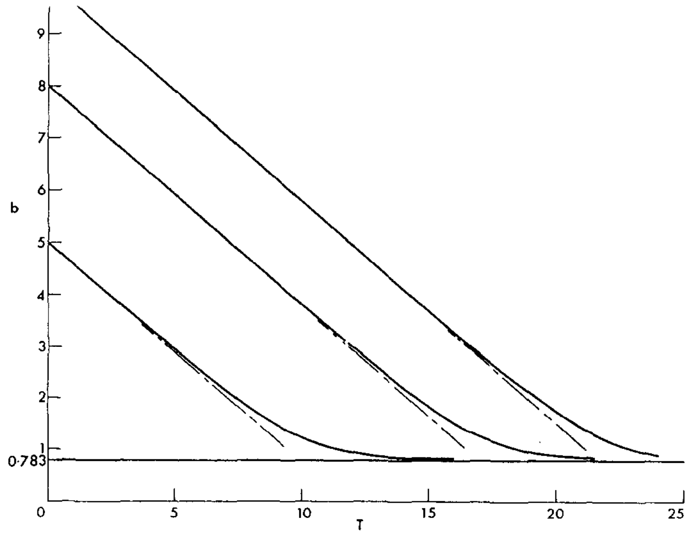


Fig. 1. Non-dimensional relation between contact radius $b$ and time $T$.

In the numerical solution of eqn. (16) the computational effort increases for each time increment and solutions extending from a large initial radius to a suitable level of convergence are very lengthy. This problem was to some extent overcome by using a variable time increment which was geared to the instantaneous contact radius.

## 9. Conclusions

Equation (16) and the numerical results of Fig. 1 constitute a solution to the transient problem of an elastic conducting sphere sliding on a rigid non-conducting plane, subject to a Hertzian approximation to the contact pressure distribution.

The non-dimensional form of the equation shows that all physical conditions are described by one parameter, the ratio $b(0)$ between the initial contact radius and the contact radius $a_0$ achieved in the steady state. If this ratio is sufficiently large the initial reduction in radius is linear with time, the rate depending only upon $a_0$ and the thermal diffusivity of the sphere. Significant divergence from linearity only starts to occur when $a/a_0 < 3$.

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