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Finite element implementation of an eigenfunction solution for the contact pressure variation due to wear

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ABSTRACT

The classical eigenfunction method for the solution of contact problems involving wear is formulated in the context of the finite element method. Static reduction is used to reduce the full stiffness matrix to the N contact nodes, after which the assumption of a separated variable solution leads to a linear eigenvalue problem with N eigenvalues and eigenfunctions. A general solution to the transient problem can then be written as an eigenfunction series, with the unknown coefficients being determined from the initial conditions.

1. Introduction

The contact pressure between two sliding bodies is influenced by the profile of the bodies, but this is altered by wear so that the pressure distribution evolves with time. If the applied loads are independent of time, generally there will exist some steady state in which further wear simply causes a change in profile that can be accommodated by a rigid-body displacement [1,2]. For example, in the rudimentary braking system shown in Fig. 1, long-term wear must be proportional to the distance x from the pivot and hence, if Archard's wear law is assumed [3], the steady-state contact pressure distribution must also be proportional to x.

Most wearing systems will start from a condition that differs from that required for the steady state and hence there will be an initial 'wearing-in' period during which the contact pressure evolves with time. In the context of classical contact mechanics, techniques have been developed for solving this transient problem, using the concept of an eigenfunction expansion [4–7]. Briefly, if the steady state contact pressure distribution is $p_0(x,y)$, we postulate that the full transient solution can be written in the form

$$p(x,y,t) = p_0(x,y) + p_1(x,y,t) \tag{1}$$

where $p_1(x,y,t)$ is a transient corrective term that can be expected to decay to zero with time. Now the inhomogeneous boundary conditions are satisfied by the steady-state solution $p_0(x,y)$ ex hypothesi, so the boundary conditions satisfied by the corrective term $p_1(x,y,t)$ must be homogeneous. With these considerations in mind, we seek solutions of this homogeneous problem in the separated-variable form

$$p_1(x,y,t) = Ce^{-\lambda t} f(x,y). \tag{2}$$

If the governing equations are linear — notably if the contact area remains constant and Archard's wear law is assumed defining the wear rate as being proportional to the local pressure — the exponential factor will cancel in the resulting boundary-value problem, leading in general to an eigenvalue problem for λ with a denumerably infinite set of eigenvalues λ_n and corresponding eigenfunctions $f_n(x,y)$. We can then write a more general solution of the transient problem in the form

$$p(x,y,t) = p_0(x,y) + \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} f_n(x,y) \tag{3}$$

where the C_n are a set of arbitrary constants to be determined from the initial conditions, notably the contact pressure at time $t=0$. For the problem to be well-posed, it is necessary that the set of eigenfunctions be complete on an appropriate domain and inversion is facilitated if they are also orthogonal, a result that can sometimes be established in analytical formulations from the properties of the resulting integral equation [8].

The range of contact problems that can be treated by analytical methods is somewhat limited, and in other cases we generally must have recourse to numerical (e.g. finite element) methods [9–11].

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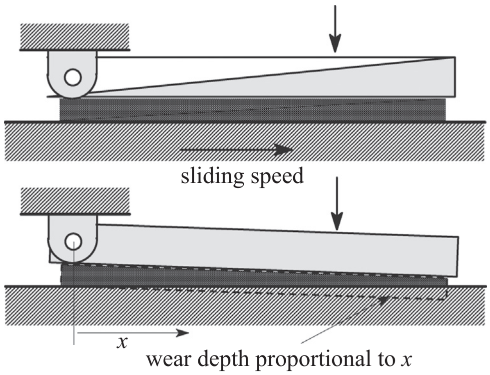

Fig. 1. Wear and hence contact pressure determined by kinematics.

These are typically performed in the time domain and can be computationally intensive, though relatively coarse time steps can be used because the geometric changes associated with wear are slow. However, if the problem is linear, we shall show that the finite element method can be used to implement the eigenfunction series method defined above. In order to ensure linearity, it is necessary that the contact area should remain constant during the wearing process and that the wear rate should be a linear function of the contact pressure.

2. Numerical formulation

The first step is to use standard static reduction (substructuring) methods to obtain the contact stiffness matrix — i.e. the linear relation between the vector of contact nodal forces and nodal displacements.

To introduce the method, we shall initially restrict attention to two-dimensional problems, for which the required operations are defined in [12], but we shall show how to generalize to three-dimensions in Section 4 below. We first develop separate finite element models of the two contacting bodies, taking care to locate the contact nodes at the same points on the interface. Each of the bodies should be supported at the points where connection is made to the rest of the braking system. For example, if the shoes and the drum are modelled for the automotive drum brake of Fig. 2, points in the flexible element should be fixed and the drum should be fixed at the connection to the axle.

We use the sign convention that the normal contact nodal force p_i is directed into the body (compressive positive) and the tangential force q_i opposes the direction of sliding motion. The corresponding normal and tangential relative nodal displacements are denoted by w_i, v_i respectively. Following the procedures defined in [12], we obtain the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ such that

$$\begin{Bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{q}^w \\ \boldsymbol{p}^w \end{Bmatrix} \tag{4}$$

where $\boldsymbol{p}^w, \boldsymbol{q}^w$ are the nodal contact forces that would be generated if relative motion at the contact nodes were prevented ($\boldsymbol{v} = \boldsymbol{w} = \boldsymbol{0}$). Notice that the full contact stiffness matrix must be symmetric and positive definite, as therefore must be the matrices $\boldsymbol{A}$ and $\boldsymbol{C}$, but no such restriction applies to the matrix $\boldsymbol{B}$ which defines the coupling between tangential forces and normal displacements and vice versa.

Eq. (4) expands as

$$\boldsymbol{q} = \boldsymbol{A}\boldsymbol{v} + \boldsymbol{B}^T\boldsymbol{w} + \boldsymbol{q}^w \tag{5}$$

$$\boldsymbol{p} = \boldsymbol{B}\boldsymbol{v} + \boldsymbol{C}\boldsymbol{w} + \boldsymbol{p}^w \tag{6}$$

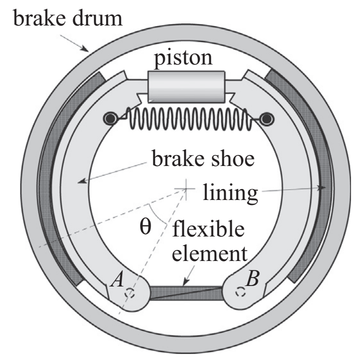

Fig. 2. An automotive drum brake.

and we can solve the first of these for $\boldsymbol{v}$ obtaining

$$\boldsymbol{v} = \boldsymbol{A}^{-1}(\boldsymbol{q} - \boldsymbol{B}^T\boldsymbol{w} - \boldsymbol{q}^w). \tag{7}$$

We might expect the tangential displacement vector $\boldsymbol{v}$ to be indeterminate to within a rigid body displacement associated with the sliding motion, but we have essentially prescribed this motion by imposing fixed support conditions on both sliding bodies. It follows that the matrix $\boldsymbol{A}$ is non-singular and hence invertible.

Substitution in Eq. (6) then yields

$$\boldsymbol{p} = \boldsymbol{B}\boldsymbol{A}^{-1}(\boldsymbol{q} - \boldsymbol{B}^T\boldsymbol{w} - \boldsymbol{q}^w) + \boldsymbol{C}\boldsymbol{w} + \boldsymbol{p}^w. \tag{8}$$

We assume that there exists an initial gap vector $\boldsymbol{g}_0$ so that under load, after some period of sliding, the gap will be

$$\boldsymbol{g} = \boldsymbol{g}_0 + \boldsymbol{w} + \boldsymbol{\delta} = \boldsymbol{0}, \tag{9}$$

where $\boldsymbol{\delta}$ defines the total nodal wear. We also assume that the Coulomb friction law applies, and hence

$$\boldsymbol{q} = f\boldsymbol{p}, \tag{10}$$

where f is the coefficient of friction.

Eliminating $\boldsymbol{q}, \boldsymbol{w}$ between Eqs. (8)–(10), we obtain

$$\boldsymbol{M}_1\boldsymbol{p} = -\boldsymbol{M}_2(\boldsymbol{\delta} + \boldsymbol{g}_0) + \boldsymbol{p}^w - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{q}^w, \tag{11}$$

where

$$\boldsymbol{M}_1 = \boldsymbol{I} - f\boldsymbol{B}\boldsymbol{A}^{-1}; \quad \boldsymbol{M}_2 = \boldsymbol{C} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^T. \tag{12}$$

2.1. Archard's wear law

We also assume that Archard's wear law applies, such that the rate of wear is proportional to the local contact pressure and the sliding velocity [3]. This concept was originally proposed by Reye [13]. A compendious collection of wear data was compiled by Lim and Ashby [14] and presented in the form of wear maps. However p_i is the nodal *force* at node i, so to implement the wear law, we must first determine the nodal contact pressure as p_i/A_i where A_i is the area of the contact surface associated with node i, typically evaluated as the average of the surface area of the elements for which i is a common node. We can then express Archard's wear law in the form

$$\dot{\boldsymbol{\delta}} = \alpha \boldsymbol{L}\boldsymbol{p} \quad \text{where } L_{ij} = \frac{V_i \delta_{ij}}{A_j}, \tag{13}$$

$\alpha$is a wear constant, V_i is the sliding velocity at node i, and δ_{ij} is the Kronecker delta.

Differentiating Eq. (11) with respect to time, we have

$$\boldsymbol{M}_1\dot{\boldsymbol{p}} = -\boldsymbol{M}_2\dot{\boldsymbol{\delta}}, \tag{14}$$

since $\boldsymbol{p}^w, \boldsymbol{q}^w, \boldsymbol{g}_0$ are independent of time. Using Archard's equation (13), this then gives

$$\boldsymbol{M}_1\dot{\boldsymbol{p}} = -\alpha \boldsymbol{M}_2\boldsymbol{L}\boldsymbol{p}, \tag{15}$$

2.2. Eigenfunction expansion

Following Eq. (2), we now seek solutions of Eq. (15) of the form

$$\boldsymbol{p} = \boldsymbol{p}_n \exp(-\lambda_n t). \tag{16}$$

This function will satisfy Eq. (15) if

$$\alpha \boldsymbol{M}_2 \boldsymbol{L} \boldsymbol{p}_n = \lambda_n \boldsymbol{M}_1 \boldsymbol{p}_n, \tag{17}$$

which defines a generalized eigenvalue problem for the eigenvalues λ_n and the corresponding eigenfunctions $\boldsymbol{p}_n$. Once these have been obtained, the general solution can be written as an eigenfunction series

$$\boldsymbol{p}(t) = \sum_{n=1}^{N} G_n \boldsymbol{p}_n \exp(-\lambda_n t), \tag{18}$$

where G_n are a set of arbitrary constants and N is the number of contact nodes. Physically, each term in this equation with $\lambda_n \neq 0$ represents a perturbation from the steady state that decays with time t, such that the most persistent perturbations correspond to the lowest values of λ_n.

2.3. Initial conditions

At time $t=0$, there is no wear ($\boldsymbol{\delta} = \boldsymbol{0}$), so Eq. (11) reduces to

$$\boldsymbol{M}_1 \boldsymbol{p}_0 = -\boldsymbol{M}_2 \boldsymbol{g}_0 + \boldsymbol{p}^w - \boldsymbol{B}\boldsymbol{A}^{-1} \boldsymbol{q}^w, \tag{19}$$

which can be solved to determine the initial value of the vector $\boldsymbol{p}(0) = \boldsymbol{p}_0$. Eq. (18) with $t=0$ then gives

$$\sum_{n=1}^{N} G_n \boldsymbol{p}_n = \boldsymbol{p}_0 \tag{20}$$

which can be solved for the constants G_n. Unfortunately, the matrix $(\boldsymbol{I} - f\boldsymbol{B}\boldsymbol{A}^{-1})$ is not symmetric, so we cannot assume that the eigenfunctions are orthogonal. However, if we define the matrix of eigenfunctions D_{ji} such that the ith eigenfunction is the ith column of $\boldsymbol{D}$, we have

$$\boldsymbol{D}\boldsymbol{G} = \boldsymbol{p}_0 \tag{21}$$

and the matrix $\boldsymbol{D}$ is non-singular since the eigenvectors must be linearly independent. We can therefore invert it, obtaining

$$\boldsymbol{G} = \boldsymbol{D}^{-1} \boldsymbol{p}_0. \tag{22}$$

2.4. Steady state

If the matrix $\boldsymbol{M}_2$ is singular, Eq. (17) will have one or more zero eigenvalues, indicating that there exists a non-trivial steady-state solution, since the exponential factors on the corresponding terms in the series (18) become unity. It can be determined by setting $t \to \infty$ in Eq. (18), so that all the terms *except* those corresponding to zero eigenvalues become zero. The constant multipliers on these terms are obtained from (22) in exactly the same way as the other constants, but in physical terms, they correspond to kinematically consistent wear patterns, with the coefficients determined by equilibrium considerations. Thus, there is no need to separate out the steady-state solution as in Eq. (1).

3. Variable load and speed

The analysis so far assumes that the speed $\boldsymbol{V}$ and the applied loads (defining $\boldsymbol{p}^w, \boldsymbol{q}^w$) remain constant throughout the wearing process. However, the method can be extended to cases where these parameters vary with time, provided that the speed is completely defined by the kinematics of the relative motion, as in sliding motion. Problems involving partial slip [15] would lead to deformation-dependent sliding speeds and hence to non-linearity in the governing equations.

3.1. Variable external loads

If the external loads are functions of time, the differentiation of (11) with respect to time yields the more general expression

$$\boldsymbol{M}_1 \dot{\boldsymbol{p}} = -\boldsymbol{M}_2 \dot{\boldsymbol{\delta}} + \dot{\boldsymbol{p}}^w - \boldsymbol{B}\boldsymbol{A}^{-1} \dot{\boldsymbol{q}}^w. \tag{23}$$

We first expand the instantaneous contact pressure vector $\boldsymbol{p}(t)$ as an eigenfunction series obtaining

$$\boldsymbol{p}(t) = \boldsymbol{D}\boldsymbol{G}(t), \tag{24}$$

where the vector $\boldsymbol{G}$ is now time-dependent, but $\boldsymbol{D}$ is time-independent, being still the matrix of eigenvectors of Eq. (17). Substituting this expression into Eq. (23), we have

$$\begin{aligned} \boldsymbol{M}_1 \boldsymbol{D} \dot{\boldsymbol{G}}(t) &= -\alpha \boldsymbol{M}_2 \boldsymbol{L} \boldsymbol{D} \boldsymbol{G}(t) + \dot{\boldsymbol{p}}^w - \boldsymbol{B}\boldsymbol{A}^{-1} \dot{\boldsymbol{q}}^w \\ &= -\boldsymbol{\Lambda} \boldsymbol{M}_1 \boldsymbol{D} \boldsymbol{G}(t) + \dot{\boldsymbol{p}}^w - \boldsymbol{B}\boldsymbol{A}^{-1} \dot{\boldsymbol{q}}^w, \end{aligned}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix of eigenvalues such that $\Lambda_{ij} = \delta_{ij}\lambda_j$ and we have used (17) to simplify the first term on the right-hand side. It follows that

$$\dot{\boldsymbol{G}}(t) + \boldsymbol{\Lambda}\boldsymbol{G}(t) = (\boldsymbol{M}_1 \boldsymbol{D})^{-1} (\dot{\boldsymbol{p}}^w - \boldsymbol{B}\boldsymbol{A}^{-1} \dot{\boldsymbol{q}}^w) \tag{25}$$

which defines a set of N uncoupled first order ODEs for the components of the time-varying vector $\boldsymbol{G}(t)$. These equations can be solved analytically for any time-varying loads $\boldsymbol{p}^w(t), \boldsymbol{q}^w(t)$. More details of this method of solution of linear time-evolution problems is given by Zagrodzki [16] who used it to solve problems involving frictionally excited thermoelastic instability.

3.2. Variable speed

Archard's wear law implies that the wear volume is proportional to the distance of sliding. Thus if we define a new time-like parameter that measures progress in the kinematic degree of freedom that permits sliding, the actual rate at which the resulting trajectory is traversed will have no effect on the solution.

For example, in a disc brake, the sliding velocity at a given point in the contact area will be $V(t) = R\Omega(t)$, where R is the distance from the axis of rotation and $\Omega(t)$ is the angular velocity which may vary with time. We define the angle of rotation of the disc $\phi(t)$ as an appropriate time-like parameter, so that

$$\Omega(t) = \dot{\phi}; \quad V(t) = R\dot{\phi}; \quad \phi = \int \Omega \, dt \tag{26}$$

and hence

$$L_{ij} = \dot{\phi} \frac{R_i \delta_{ij}}{A_j}, \tag{27}$$

from Eq. $(13)_2$. Archard's wear law $(13)_1$ can then be written as

$$\dot{\boldsymbol{\delta}} = \dot{\phi} \frac{d\boldsymbol{\delta}}{d\phi} = \alpha \dot{\phi} \boldsymbol{L}^{(\phi)} \boldsymbol{p} \tag{28}$$

where $L_{ij}^{(\phi)} = R_i \delta_{ij} / A_j$. It follows that

$$\frac{d\boldsymbol{\delta}}{d\phi} = \alpha \boldsymbol{L}^{(\phi)} \boldsymbol{p}, \tag{29}$$

and the resulting eigenfunction solution of this equation (in the ϕ-domain) is independent of any variation in rotational speed. Transformation into wear and contact pressure as functions of time is then readily accomplished, using the relations (26).

4. Three-dimensional problems

If the problem is three-dimensional, we can resolve the contact traction at node i into a normal (compressive) traction p_i, a tangential traction q_i in the direction opposing the sliding motion, and a tangential traction r_i orthogonal to the sliding direction. We also identify the corresponding contact displacements as w_i, v_i, u_i respectively. Static reduction then permits a statement of the elasticity problem in the form

$$\begin{Bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} & \boldsymbol{K}_{13} \\ \boldsymbol{K}_{12}^T & \boldsymbol{K}_{22} & \boldsymbol{K}_{23} \\ \boldsymbol{K}_{13}^T & \boldsymbol{K}_{23}^T & \boldsymbol{K}_{33} \end{bmatrix} \begin{Bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \\ \boldsymbol{u} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{p}^0 \\ \boldsymbol{q}^0 \\ \boldsymbol{r}^0 \end{Bmatrix}, \tag{30}$$

where $\boldsymbol{p}^0, \boldsymbol{q}^0, \boldsymbol{r}^0$ are the nodal contact forces that would be generated by the external forces if the contact nodes were prevented from moving — i.e. if $\boldsymbol{w} = \boldsymbol{v} = \boldsymbol{u} = \boldsymbol{0}$.

4.1. Reduction to a two-dimensional problem

We assume that the bodies are restrained from relative rigid-body motion in the direction orthogonal to the sliding direction, so the matrix $\boldsymbol{K}_{33}$ is non-singular. The tangential force must everywhere oppose the direction of sliding, which implies that $\boldsymbol{r} = \boldsymbol{0}$, so we can use this result to eliminate the lateral displacement component $\boldsymbol{u}$, obtaining

$$\boldsymbol{u} = -[\boldsymbol{K}_{33}]^{-1}(\boldsymbol{r}^0 + \boldsymbol{K}_{13}^T \boldsymbol{w} + \boldsymbol{K}_{23}^T \boldsymbol{v}). \tag{31}$$

Substitution into (30) then yields an equation identical to (4) with

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{K}_{22} - \boldsymbol{K}_{23}[\boldsymbol{K}_{33}]^{-1}\boldsymbol{K}_{23}^T \\ \boldsymbol{B} &= \boldsymbol{K}_{12} - \boldsymbol{K}_{13}[\boldsymbol{K}_{33}]^{-1}\boldsymbol{K}_{23}^T \\ \boldsymbol{C} &= \boldsymbol{K}_{11} - \boldsymbol{K}_{13}[\boldsymbol{K}_{33}]^{-1}\boldsymbol{K}_{13}^T \end{aligned} \tag{32}$$

and

$$\begin{aligned} \boldsymbol{p}^w &= \boldsymbol{p}^0 - \boldsymbol{K}_{13}[\boldsymbol{K}_{33}]^{-1}\boldsymbol{r}^0 \\ \boldsymbol{q}^w &= \boldsymbol{q}^0 - \boldsymbol{K}_{23}[\boldsymbol{K}_{33}]^{-1}\boldsymbol{r}^0. \end{aligned} \tag{33}$$

The solution can then be obtained as in Section 2.

5. Example

To illustrate the use of the method, we created a finite element model of a commercial automotive drum brake. For simplicity, this was converted to a two-dimensional problem, the mesh being shown in Fig. 3. Since the actual shoe has a T-section, the elastic modulus in the vertical part of the T was reduced in the two-dimensional model to compensate for the reduced thickness.

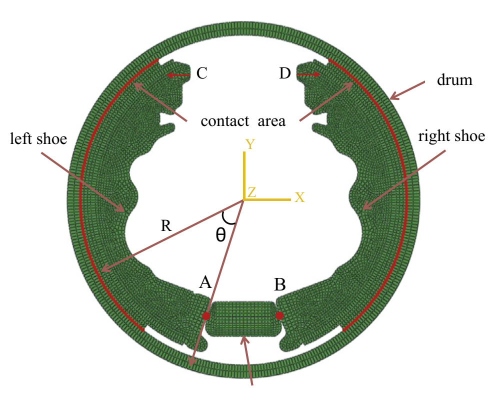

Fig. 3. Finite element mesh for the drum brake.

The flexible element was fixed at the points A, B and the brake shoes were loaded by equal and opposite forces at C, D. The complete stiffness matrix was exported from Abaqus and the reduced matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and the nodal loading vectors $\boldsymbol{p}^w, \boldsymbol{q}^w$ were determined as in [12]. The flexible element exerts a very small restraint against rigid-body rotation, so the resulting eigenvalue problem (17) does not exhibit a zero eigenvalue. However, the lowest eigenvalue is two orders of magnitude smaller than the next highest, showing that the system mimics the behaviour of a shoe with a frictionless pivot.

Fig. 4 shows the evolution of the contact pressure distribution at the left brake shoe, determined from Eqs. (18) to (20) as functions of the dimensionless time

$$\tau = \frac{\alpha V E t}{R}, \tag{34}$$

where E is Young's modulus and R is the radius of the contact surface. Notice that the pressure evolves from the initial condition to an almost time invariant distribution for $\tau > 10$. Corresponding results for the right shoe are shown in Fig. 5 and exhibit significantly lower contact pressures. This occurs because the frictional forces enhance the moment due to the external loading on the left shoe but detract from it on the right shoe.

6. Conclusions

Static reduction of the finite element stiffness matrix followed by an eigenvalue formulation of the evolutionary wear problem defines an extremely efficient method for tracking the changes in contact pressure due to wear in sliding systems. The method is restricted to linear systems for which full contact is maintained at all times, but it can be applied to three-dimensional problems and to problems in which the applied load and speed vary with time.

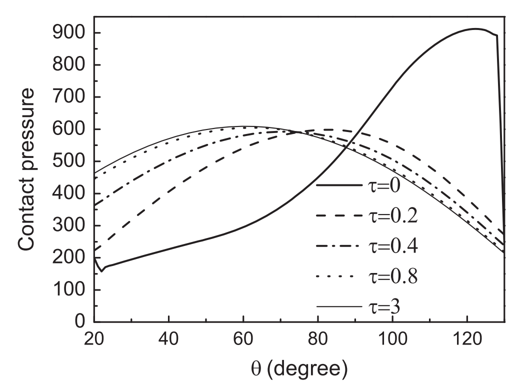

Fig. 4. Evolution of contact pressure for the left shoe.

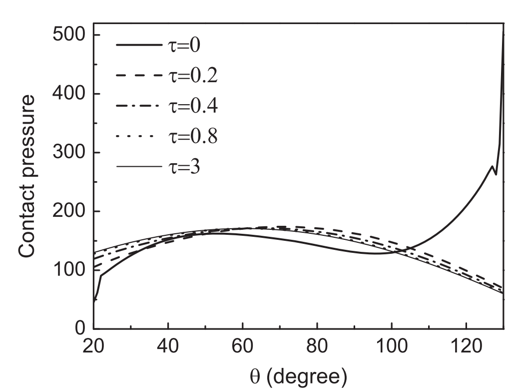

Fig. 5. Evolution of contact pressure for the right shoe.

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