THERMAL PROBLEMS WITH RADIATION BOUNDARY CONDITIONS

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SUMMARY

Contact or crack problems in thermoelasticity are usually analysed with the idealised boundary conditions of perfect conduction or perfect insulation. These boundary conditions, while simplifying the mathematics, sometimes lead to unrealistic, singular thermoelastic fields. This paper formulates axisymmetric static *thermal* problems for a half-space when one boundary condition corresponds to partial insulation, either inside or outside the circle $r = a$, $z = 0$. Four important cases are considered and the problems are reduced to the solution of integro-differential equations of Abel type. In each case it is shown that the equation can be solved by using *two simultaneous Fourier expansions* of the unknown function.

1. Introduction

CONTACT and crack problems in static thermoelasticity are usually solved under the assumption that the temperature is prescribed on one part of the boundary and the heat flux on the remainder. This assumption is made to simplify the mathematics. When, for instance, the thermoelastic problem can be divided into a purely thermal problem and an elastic one (see, e.g., Olesiak and Sneddon (1), George and Sneddon (2)), then the thermal field can often be written down in explicit form without the necessity of solving an integral equation. However, there is an accompanying disadvantage in that the resulting thermoelastic field can have singular properties. For instance, in the simple problem of a uniform heat source q_0 applied over the portion $x^2 + y^2 \leq a^2$ of the otherwise insulated boundary $z = 0$ of the half-space $z \geq 0$, then (3) the normal thermoelastic displacement on the boundary outside the circle $r \leq a$ is proportional to $\ln(r/a)$, and so is *infinite* at infinity. This singular behaviour at infinity can be removed by changing the boundary condition outside the circle in such a way that the total heat flux

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entering the half-space is zero. It is shown that one way to do this is to replace the *insulated* condition by the more realistic *radiation* condition.

The physical aspects of this problem are the subject of the sequel. The purpose of the present paper is to formulate mixed boundary-value problems for the static temperature field under radiation conditions and to reduce them to integral equations that are suitable for computation.

In order to reduce the analytical complications, attention will be directed to axisymmetric problems for the half-space $z \geq 0$. The basic problem is to find the harmonic temperature field $T(r, z)$ in cylindrical polar coordinates (r, z) , satisfying the conditions (Carslaw and Jaeger (5))

$$k_1 \frac{\partial T}{\partial z} - h_1 T = R_1(r), \quad r < a, \quad z = 0, \quad (1.1)$$

$$k_2 \frac{\partial T}{\partial z} - h_2 T = R_2(r), \quad r > a, \quad z = 0. \quad (1.2)$$

The functions $R_1(r)$, $R_2(r)$ are supposed to be specified, but must be such that $T(r, z)$ is continuous, and $T(r, z) \rightarrow 0$ as $r^2 + z^2 \rightarrow \infty$.

Although it is possible to solve the mixed boundary-value problem as stated, it seemed to us to be useful to confine attention to the four limiting cases which appear in practice, in which one of the four quantities $h_1 a$, $h_2 a$, k_1 , k_2 becomes so small compared to the other three that it may be neglected.

Before proceeding to the analysis, we may make two other simplifications. Introduce dimensionless coordinates (ρ, ζ) such that $r = a\rho$, $z = a\zeta$, and the dimensionless Biot numbers $h_i a/k_i = H_i$, $i = 1, 2$; then one limiting case of equations (1.1), (1.2) corresponding to $k_2 \rightarrow 0$, is

$$\frac{\partial T}{\partial \zeta} - H_1 T = R(\rho), \quad \rho < 1, \quad z = 0, \quad (1.3)$$

$$T = T_2(\rho), \quad \rho > 1, \quad z = 0, \quad (1.4)$$

where $H_1 = h_1 a/k_1$, $R(\rho) = aR_1(r)/k_1$.

Let S , S' denote the regions $\rho < 1$ and $\rho > 1$ respectively on the surface of the half-space. The four basic physical problems are:

- (i) S is partially insulated while S' is completely conducting and held at constant temperature. On $\zeta = 0$,

$$\frac{\partial T}{\partial \zeta} - H_1 T = R(\rho), \quad \rho < 1; \quad T = 0, \quad \rho > 1. \quad (1.5)$$

- (ii) S is partially insulated while S' is completely insulated. On $\zeta = 0$,

$$\frac{\partial T}{\partial \zeta} - H_1 T = R(\rho), \quad \rho < 1; \quad \frac{\partial T}{\partial \zeta} = 0, \quad \rho > 1. \quad (1.6)$$

- (iii) S is kept at a specified temperature while S' is partially insulated. On $\zeta = 0$,

$$T = T_0, \quad \rho < 1; \quad \frac{\partial T}{\partial \zeta} - H_2 T = 0, \quad \rho > 1. \quad (1.7)$$

- (iv) S is subjected to a specified heat flux while S' is partially insulated. On $\zeta = 0$,

$$\frac{\partial T}{\partial \zeta} = F_0, \quad \rho < 1; \quad \frac{\partial T}{\partial \zeta} - H_2 T = 0, \quad \rho > 1. \quad (1.8)$$

In each case $\partial T / \partial \zeta = \partial T(\rho, 0) / \partial \zeta|_{\zeta=0}$, $T \equiv T(\rho, 0)$ and $R(\rho)$ must satisfy regularity conditions so that $T(\rho, \zeta) \rightarrow 0$ as $\rho^2 + \zeta^2 \rightarrow \infty$.

It was noted in the introduction that the solutions of problems (iii) and (iv) have the important property that the total flux of heat into the half-space is zero, i.e.

$$\int_0^\infty \rho \frac{\partial T}{\partial \zeta}(\rho, 0) d\rho = 0. \quad (1.9)$$

This property is proved later.

2. Preliminary analysis

The analysis follows similar lines in each of the four cases. In each case the harmonic function $T(\rho, \zeta)$ is represented as a Hankel transform

$$T(\rho, \zeta) = \mathcal{H}_0[\xi^{-1} A(\xi) \exp(-\xi \zeta); \rho], \quad (2.1)$$

and extensive use is made of the relationships between Hankel, Fourier cosine, Fourier sine and Abel transforms. These are defined as follows (see (5) or (6), chapter 5):

$$\text{Hankel:} \quad \mathcal{H}_\nu[\xi^{-1} f(\xi); \rho] = \int_0^\infty f(\xi) J_\nu(\xi \rho) d\xi; \quad (2.2)$$

$$\text{Fourier:} \quad \mathcal{F}_c[f(\xi); x] = (2/\pi)^{1/2} \int_0^\infty f(\xi) \cos(\xi x) d\xi, \quad (2.3)$$

$$\mathcal{F}_s[f(\xi); x] = (2/\pi)^{1/2} \int_0^\infty f(\xi) \sin(\xi x) d\xi; \quad (2.4)$$

$$\text{Abel:} \quad \mathcal{A}_1[f(x); \rho] = (2/\pi)^{1/2} \int_0^\rho (\rho^2 - x^2)^{-1/2} f(x) dx, \quad (2.5)$$

$$\mathcal{A}_2[f(x); \rho] = (2/\pi)^{1/2} \int_\rho^\infty (x^2 - \rho^2)^{-1/2} f(x) dx, \quad (2.6)$$

with inverses

$$\mathcal{A}_1^{-1}[F(\rho); x] = D_x \mathcal{A}_1[\rho F(\rho); x], \quad (2.7)$$

$$\mathcal{A}_2^{-1}[F(\rho); x] = -D_x \mathcal{A}_2[\rho F(\rho); x], \quad (2.8)$$

where $D_x = d/dx$.

The following relationships are known (6, chapter 5):

$$\mathcal{A}_1^{-1}\{\mathcal{H}_0[\xi^{-1}A(\xi); \rho]; x\} = \mathcal{F}_c[A(\xi); x], \quad (2.9)$$

$$\mathcal{A}_1\{\rho\mathcal{H}_0[A(\xi); \rho]; x\} = \mathcal{F}_s[A(\xi); x], \quad (2.10)$$

$$\mathcal{A}_2^{-1}\{\mathcal{H}_0[\xi^{-1}A(\xi); \rho]; x\} = \mathcal{F}_s[A(\xi); x], \quad (2.11)$$

$$\mathcal{A}_2\{\rho\mathcal{H}_0[A(\xi); \rho]; x\} = \mathcal{F}_c[A(\xi); x]. \quad (2.12)$$

In the solution of the integral equations extensive use is made of Legendre polynomials, and in particular the following results are used (see (7), 3.675):

$$\int_{\phi}^{\frac{1}{2}\pi} \frac{\sin(2n+1)\theta \, d\theta}{\{2(\cos 2\phi - \cos 2\theta)\}^{\frac{1}{2}}} = \frac{1}{4}\pi P_n(\cos 2\phi), \quad (2.13)$$

$$\int_0^{\phi} \frac{\cos(2n+1)\theta \, d\theta}{\{2(\cos 2\theta - \cos 2\phi)\}^{\frac{1}{2}}} = \frac{1}{4}\pi P_n(\cos 2\phi), \quad (2.14)$$

$$\frac{P_n(x) - P_{n+2}(x)}{1-x^2} = \frac{(2n+3)}{2(n+1)(n+2)} \sum_{k=0}^n \{1 + (-)^{n+k}\} (2k+1) P_k(x). \quad (2.15)$$

3. Case (i)

The boundary conditions on $T(\rho, \zeta)$ are (1.5), namely

$$\frac{\partial T}{\partial \zeta} - HT = R(\rho), \quad \rho < 1; \quad T = 0, \quad \rho > 1, \quad (3.1)$$

so that the Hankel form (2.1) yields the dual integral equations

$$\mathcal{H}_0[A(\xi); \rho] + H\mathcal{H}_0[\xi^{-1}A(\xi); \rho] = -R(\rho), \quad \rho < 1, \quad (3.2)$$

$$\mathcal{H}_0[\xi^{-1}A(\xi); \rho] = 0, \quad \rho > 1. \quad (3.3)$$

Apply $\mathcal{A}_2^{-1}$ to equation (3.3) and use (2.11) to obtain

$$\mathcal{F}_s[A(\xi); x] = (\tfrac{1}{2}\pi)^{\frac{1}{2}} f(x) U(1-x), \quad (3.4)$$

where U is the Heaviside unit function. Now express equation (3.2) in terms of $f(x)$. Equations (2.10), (2.11) give

$$\begin{aligned} \mathcal{H}_0[A(\xi); \rho] &= \frac{1}{\rho} \mathcal{A}_1^{-1}\{\mathcal{F}_s[A(\xi); x]; \rho\} \\ &= \frac{1}{\rho} \frac{d}{d\rho} \int_0^{\rho} \frac{xf(x) \, dx}{(\rho^2 - x^2)^{\frac{1}{2}}}, \quad \rho < 1, \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \mathcal{H}_0[\xi^{-1}A(\xi); \rho] &= \mathcal{A}_2\{\mathcal{F}_s[A(\xi); x]; \rho\} \\ &= \int_{\rho}^1 \frac{f(x) \, dx}{(x^2 - \rho^2)^{\frac{1}{2}}}, \quad \rho < 1, \end{aligned} \quad (3.6)$$

so that

$$\frac{1}{\rho} \frac{d}{d\rho} \int_0^\rho \frac{xf(x) dx}{(\rho^2 - x^2)^{\frac{1}{2}}} + H \int_\rho^1 \frac{f(x) dx}{(x^2 - \rho^2)^{\frac{1}{2}}} = -R(\rho), \quad \rho < 1. \quad (3.7)$$

In order to reduce this equation to a suitable form for computation we put $x = \cos \theta$, $\rho = \cos \phi$ and use *two simultaneous representations* for $f(x)$, namely

$$f(x) = F(\theta) = \sum_{n=0}^{\infty} a_n \cos(2n+1)\theta, \quad 0 < \theta < \frac{1}{2}\pi, \quad (3.8)$$

$$\sin \theta F(\theta) = \sum_{n=0}^{\infty} b_n \cos(2n+1)\theta, \quad 0 < \theta < \frac{1}{2}\pi, \quad (3.9)$$

Then on using (2.13), (2.14) we find that

$$\begin{aligned} \int_0^\rho \frac{xf(x) dx}{(\rho^2 - x^2)^{\frac{1}{2}}} &= \sum_{n=0}^{\infty} {}^* a_n \int_\phi^\pi \frac{\sin 2\theta \cos(2n+1)\theta d\theta}{\{2(\cos 2\phi - \cos 2\theta)\}^{\frac{1}{2}}} \\ &= \frac{\pi}{8} \sum_{n=0}^{\infty} {}^* a_n \{P_{n+1}(\cos 2\phi) - P_{n-1}(\cos 2\phi)\}, \end{aligned} \quad (3.10)$$

where the * indicates that the P_{-1} occurring when $n=0$ is taken to be P_0 . Similarly

$$\begin{aligned} \int_\rho^1 \frac{f(x) dx}{(x^2 - \rho^2)^{\frac{1}{2}}} &= 2 \sum_{n=0}^{\infty} b_n \int_0^\phi \frac{\cos(2n+1)\theta d\theta}{\{2(\cos 2\phi - \cos 2\theta)\}^{\frac{1}{2}}} \\ &= \frac{\pi}{2} \sum_{n=0}^{\infty} b_n P_n(\cos 2\phi). \end{aligned} \quad (3.11)$$

Now equation (2.15) and the rule for differentiation of a Legendre polynomial yield

$$\frac{1}{\rho} \frac{d}{d\rho} \{P_{n+1}(2\rho^2 - 1) - P_{n-1}(2\rho^2 - 1)\} = 4(2n+1)P_n(2\rho^2 - 1). \quad (3.12)$$

This holds also, when the above mentioned convention is used, for $n=0$, so that equation (3.7) becomes

$$\frac{\pi}{2} \sum_{n=0}^{\infty} (2n+1)a_n P_n(2\rho^2 - 1) + \frac{\pi}{2} H \sum_{n=0}^{\infty} b_n P_n(2\rho^2 - 1) = -R(\rho). \quad (3.13)$$

Thus if we expand $R(\rho)$ in the form

$$R(\rho) = \frac{\pi}{2} \sum_{n=0}^{\infty} (2n+1)g_n P_n(2\rho^2 - 1), \quad (3.14)$$

then we have

$$a_n + H(2n+1)^{-1}b_n = -g_n, \quad n = 0, 1, 2, \dots \quad (3.15)$$

This must be supplemented by the relationship between a_n and b_n , namely

$$b_n = \sum_{m=0}^{\infty} c_{m,n} a_m, \quad (3.16)$$

where

$$\begin{aligned} c_{m,n} &= \frac{2}{\pi} \int_0^{\pi} \sin \theta \cos (2m+1)\theta \cos (2n+1)\theta d\theta \\ &= \frac{1}{\pi} \left(\frac{1}{2n+2m+3} - \frac{1}{2n+2m+1} + \frac{1}{2n-2m+1} - \frac{1}{2n-2m-1} \right). \end{aligned} \quad (3.17)$$

It may be shown that the set of equations (3.15), (3.16) is *regular* in the sense of Kantorovitch and Krylov (8); they may therefore be solved by truncation.

Collins (9) and Barber (10) reduced the problem (3.1) to the integral equation

$$-\frac{H}{\pi} \int_{-1}^1 \ln |t-x| f(t) dt + f(x) = -\frac{2}{\pi} \int_0^x \frac{\rho R(\rho) d\rho}{(x^2 - \rho^2)^{1/2}}. \quad (3.18)$$

This is seen to yield equations (3.15), (3.16) by noting that

$$\begin{aligned} -\frac{1}{\pi} \int_{-1}^1 \ln |t-x| f(t) dt &= -\frac{1}{\pi} \int_0^{\pi} \ln |\cos \theta - \cos \phi| F(\theta) \sin \theta d\theta \\ &= -\frac{1}{\pi} \sum_{n=0}^{\infty} b_n \int_0^{\pi} \ln |\cos \theta - \cos \phi| \cos (2n+1)\theta d\theta \\ &= \sum_{n=0}^{\infty} (2n+1)^{-1} b_n \cos (2n+1)\phi, \end{aligned} \quad (3.19)$$

and that equation (3.14) yields

$$\frac{2}{\pi} \int_0^x \frac{\rho R(\rho) d\rho}{(x^2 - \rho^2)^{1/2}} = \sum_{n=0}^{\infty} g_n \cos (2n+1)\phi. \quad (3.20)$$

4. Case (ii)

The boundary conditions are

$$\frac{\partial T}{\partial \xi} - HT = R(\rho), \quad \rho < 1; \quad \frac{\partial T}{\partial \xi} = 0, \quad \rho > 1, \quad (4.1)$$

so that the Hankel forms yield the equations

$$\mathcal{H}_0[A(\xi); \rho] + H\mathcal{H}_0[\xi^{-1}A(\xi); \rho] = -R(\rho), \quad \rho < 1, \quad (4.2)$$

$$\mathcal{H}_0[A(\xi); \rho] = 0, \quad \rho > 1. \quad (4.3)$$

Apply $\mathcal{A}_2\rho$ to (4.3) and use (2.13) to obtain

$$\mathcal{F}_c[A(\xi); x] = (\tfrac{1}{2}\pi)^{1/2} f(x) U(1-x), \quad (4.4)$$

which, when substituted into (4.2), yields the Abel equation

$$-\frac{1}{\rho} \frac{d}{d\rho} \int_{\rho}^1 \frac{xf(x) dx}{(x^2 - \rho^2)^{\frac{1}{2}}} + H \int_0^{\rho} \frac{f(x) dx}{(\rho^2 - x^2)^{\frac{1}{2}}} = -R(\rho), \quad \rho < 1. \quad (4.5)$$

The appropriate representations for $f(x)$ are now

$$f(x) = F(\theta) = \sum_{n=0}^{\infty} a_n \sin(2n+1)\theta, \quad (4.6)$$

$$\sin \theta F(\theta) = \sum_{n=0}^{\infty} b_n \sin(2n+1)\theta, \quad (4.7)$$

for then

$$-\frac{1}{\rho} \frac{d}{d\rho} \int_{\rho}^1 \frac{xf(x) dx}{(x^2 - \rho^2)^{\frac{1}{2}}} = \frac{\pi}{2} \sum_{n=0}^{\infty} (2n+1) a_n P_n(2\rho^2 - 1), \quad (4.8)$$

$$\int_0^{\rho} \frac{f(x) dx}{(\rho^2 - x^2)^{\frac{1}{2}}} = \frac{\pi}{2} \sum_{n=0}^{\infty} b_n P_n(2\rho^2 - 1), \quad (4.9)$$

and equation (4.5) again yields (3.15). The relationship between the a_n and b_n is now

$$b_n = \sum_{m=0}^{\infty} d_{m,n} a_m, \quad (4.10)$$

where

$$\begin{aligned} d_{m,n} &= \frac{2}{\pi} \int_0^{\pi} \sin \theta \sin(2n+1)\theta \sin(2m+1)\theta d\theta \\ &= \frac{1}{\pi} \left(\frac{1}{2n-2m+1} - \frac{1}{2n-2m-1} - \frac{1}{2n+2m+3} + \frac{1}{2n+2m+1} \right). \end{aligned} \quad (4.11)$$

We note that there are two other equivalent integral equations for $f(x)$, namely

$$\int_{-1}^1 \left\{ \frac{1}{\pi(t-x)} + \frac{H}{2} \operatorname{sgn}(t-x) \right\} f(t) dt = \frac{2}{\pi} \int_0^x \frac{\rho R(\rho) d\rho}{(x^2 - \rho^2)^{\frac{1}{2}}}, \quad (4.12)$$

and

$$f(x) + \frac{H}{\pi} \int_0^1 f(y) \ln \left| \frac{(1-x^2)^{\frac{1}{2}} + (1-y^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}} - (1-y^2)^{\frac{1}{2}}} \right| dy = -\frac{2}{\pi} \int_x^1 \frac{\rho R(\rho) d\rho}{(\rho^2 - x^2)^{\frac{1}{2}}}. \quad (4.13)$$

Equation (4.12) appears in load-transfer problems and has been studied by Morar' and Popov (11), Arutiunian and Mkhitarian (12), and Erdogan and Gupta (13). Equations (4.13) may be solved by using the expansion

$$\ln \left| \frac{\sin \theta + \sin \phi}{\sin \theta - \sin \phi} \right| = 4 \sum_{m=0}^{\infty} (2m+1)^{-1} \sin(2m+1)\theta \sin(2m+1)\phi. \quad (4.14)$$

With the representations (4.6), (4.7), both (4.12) and (4.13) yield equation (3.15), (4.11). It is shown in (6), chapter 7, that the set of equations (3.15), (4.11) is regular.

5. Case (iii)

The boundary conditions on $T(\rho, \zeta)$ are

$$T = 1, \quad \rho < 1; \quad \frac{\partial T}{\partial \zeta} - HT = 0, \quad \rho > 1. \quad (5.1)$$

Using the representation (2.1) we may, following Sneddon (14), reduce the problem to the solution of a Fredholm integral equation with non-singular kernel. However, the kernel involves the cosine-integral function and is not suitable for computational purposes. We therefore proceed as follows.

The field $T(\rho, \zeta)$ is continuous at $\rho = 1$, $\zeta = 0$, and satisfies the zero-flux condition (1.9). We write

$$T = T_1 + T_2 + T_3, \quad (5.2)$$

where $T_1 + T_2$ itself satisfies the continuity and zero-flux conditions. Choose T_1 , T_2 so that

$$T_1 = 1, \quad \rho < 1; \quad \frac{\partial T_1}{\partial \zeta} = 0, \quad \rho > 1, \quad (5.3)$$

$$T_2 = 0, \quad \rho < 1; \quad T_2 = F(\rho), \quad \rho > 1. \quad (5.4)$$

The solution for T_1 is well known and, in particular,

$$\frac{\partial T_1}{\partial \zeta} = -\frac{2}{\pi} (1 - \rho^2)^{-\frac{1}{2}}, \quad \rho < 1; \quad T_1 = \frac{2}{\pi} \sin^{-1} \left(\frac{1}{\rho} \right), \quad \rho > 1. \quad (5.5)$$

For $F(\rho)$ we may take

$$F(\rho) = -\frac{2}{\pi} \frac{(\rho^2 - 1)^{\frac{1}{2}}}{\rho^2}, \quad (5.6)$$

and then

$$\frac{\partial T_2}{\partial \zeta} = \frac{2}{\pi \rho^3} \{ \sin^{-1} \rho - \rho(1 - \rho^2)^{-\frac{1}{2}} \}, \quad \rho < 1; \quad \frac{\partial T_2}{\partial \zeta} = \frac{1}{\rho^3}, \quad \rho > 1. \quad (5.7)$$

The boundary conditions for T_3 are therefore

$$T_3 = 0, \quad \rho < 1, \quad (5.8)$$

$$\frac{\partial T_3}{\partial \zeta} - HT_3 = \frac{2H}{\pi} \left\{ \sin^{-1} \left(\frac{1}{\rho} \right) - \frac{(\rho^2 - 1)^{\frac{1}{2}}}{\rho^2} \right\} - \frac{1}{\rho^3} \equiv R(\rho), \quad \rho > 1. \quad (5.9)$$

Proceeding as before we use the Hankel solution to obtain

$$\mathcal{H}_0[\xi^{-1}A(\xi); \rho] = 0, \quad \rho < 1, \quad (5.10)$$

$$\mathcal{H}_0[A(\xi); \rho] + H\mathcal{H}_0[\xi^{-1}A(\xi); \rho] = -R(\rho), \quad \rho > 1. \quad (5.11)$$

Thus

$$\mathcal{F}_c[A(\xi); x] = (\tfrac{1}{2}\pi)^{\frac{1}{2}} f(x) U(x-1), \quad (5.12)$$

and hence

$$\frac{1}{\rho} \frac{d}{d\rho} \int_0^\infty \frac{x f(x) dx}{(x^2 - \rho^2)^{\frac{1}{2}}} - H \int_1^\rho \frac{f(x) dx}{(\rho^2 - x^2)^{\frac{1}{2}}} = R(\rho), \quad \rho > 1. \quad (5.13)$$

Put

$$x = 1/y, \quad \rho = 1/s, \quad f(x) = y^2 j(y); \quad (5.14)$$

then

$$s^2 \frac{d}{ds} \int_0^s \frac{s j(y) dy}{(s^2 - y^2)^{\frac{1}{2}}} + H \int_s^1 \frac{y j(y) dy}{(y^2 - s^2)^{\frac{1}{2}}} = -S(s), \quad (5.15)$$

where

$$S(s) = \frac{1}{s} R\left(\frac{1}{s}\right) = \frac{2H}{\pi} \left\{ \frac{\sin^{-1} s}{s} - (1-s^2)^{\frac{1}{2}} \right\} - s^2. \quad (5.16)$$

In order to obtain appropriate representations for $j(y)$ we put $y = \cos \theta$, $s = \cos \phi$, and first take

$$j(y) = J(\theta) = \sum_{n=0}^{\infty} \alpha_n \cos 2n\theta, \quad (5.17)$$

$$\sin \theta J(\theta) = \sum_{n=0}^{\infty} \beta_n \cos 2n\theta, \quad (5.18)$$

where the prime indicates that the first term is $\frac{1}{2}\beta_0$. Now

$$\int_0^s \frac{j(y) dy}{(s^2 - y^2)^{\frac{1}{2}}} = \frac{\pi}{2} \alpha_0 + \frac{\pi}{4} \sum_{n=1}^{\infty} \alpha_n \{P_n(2s^2 - 1) - P_{n-1}(2s^2 - 1)\}. \quad (5.19)$$

After some manipulation it is found that

$$\frac{d}{ds} \{s(P_n - P_{n-1})\} = (2n+1)P_n + (2n-1)P_{n-1}, \quad n \geq 1, \quad (5.20)$$

where $P_n \equiv P_n(2s^2 - 1)$ hereafter. Thus

$$\begin{aligned} \frac{d}{ds} \int_0^s \frac{s j(y) dy}{(s^2 - y^2)^{\frac{1}{2}}} &= \frac{\pi}{2} \alpha_0 + \frac{\pi}{4} \sum_{n=1}^{\infty} \alpha_n \{(2n+1)P_n + (2n-1)P_{n-1}\} \\ &= \frac{\pi}{4} (2\alpha_0 + \alpha_1) + \frac{\pi}{4} \sum_{n=1}^{\infty} (2n+1)(\alpha_n + \alpha_{n+1})P_n. \end{aligned} \quad (5.21)$$

In order to simplify this expression we note that if (5.17) is rewritten

$$\cos \theta J(\theta) = \sum_{n=0}^{\infty} a_n \cos (2n+1)\theta, \quad (5.22)$$

then, since

$$\frac{\cos (2n+1)\theta}{\cos \theta} = 2 \cos 2n\theta - 2 \cos (2n-2)\theta + \dots 2(-1)^{n-1} \cos 2\theta + (-1)^n, \quad (5.23)$$

we have

$$2\alpha_0 + \alpha_1 = 2a_0, \quad \alpha_n + \alpha_{n+1} = 2a_n, \quad n \geq 1, \quad (5.24)$$

so that

$$\frac{d}{ds} \int_0^s \frac{sj(y) dy}{(s^2 - y^2)^{1/2}} = \frac{\pi}{2} \sum_{n=0}^{\infty} (2n+1)a_n P_n. \quad (5.25)$$

The second integral in (5.15) is

$$\int_s^1 \frac{yj(y) dy}{(s^2 - y^2)^{1/2}} = \frac{\pi}{4} \beta_0 + \frac{\pi}{4} \sum_{n=0}^{\infty} \beta_{n+1} (P_{n+1} + P_n). \quad (5.26)$$

The sum on the right is divisible by s^2 ; in fact

$$\frac{P_{n+1} + P_n}{s^2} = \frac{2}{n+1} \{(2n+1)P_n - (2n-1)P_{n-1} + \dots + (-1)^n P_0\}, \quad (5.27)$$

so that $Q_n \equiv (P_n + P_{n-1})/2s^2$ satisfies the equations

$$Q_1 = 1, \quad nQ_n + (n+1)Q_{n+1} = (2n+1)P_n. \quad (5.28)$$

Thus, since the first term in (5.15) and $S(s)$ on the right are both divisible by s^2 , we deduce that $H\beta_0 = 0$. But

$$H\beta_0 = \frac{2H}{\pi} \int_0^\pi \sin \theta J(\theta) d\theta = \frac{2H}{\pi} \int_0^1 j(y) dy = \frac{2H}{\pi} \int_1^\infty f(x) dx, \quad (5.29)$$

so that if $H \neq 0$, then

$$\int_1^\infty f(x) dx = 0.$$

This is the zero-flux condition mentioned earlier.

In order to simplify (5.26) we use (5.28) which shows that if (5.18) is rewritten

$$\sin \theta J(\theta) = \sum_{n=0}^{\infty} (2n+1)^{-1} b_n \{n \cos 2n\theta + (n+1) \cos (2n+2)\theta\}, \quad (5.30)$$

where

$$\frac{n+1}{2n+1} b_n + \frac{n+1}{2n+3} b_{n+1} = \beta_{n+1}, \quad n = 0, 1, \dots, \quad (5.31)$$

then

$$\frac{1}{s^2} \int_s^1 \frac{y j(y) dy}{(s^2 - y^2)^{\frac{1}{2}}} = \frac{\pi}{2} \sum_{n=0}^{\infty} b_n P_n. \quad (5.32)$$

If therefore we may expand $S(s)/s^2$ in the form

$$\frac{S(s)}{s^2} = \frac{\pi}{2} \sum_{n=0}^{\infty} (2n+1) g_n P_n, \quad (5.33)$$

then equation (5.15) reduces to equation (3.15)!

In order to expand $S(s)$ in terms of the P_n we note that

$$\begin{aligned} \frac{\sin^{-1} s}{s} &= \frac{\pi}{2s} - \frac{1}{s} \left(\frac{\pi}{2} - \sin^{-1} s \right) = \int_s^1 \frac{d\rho}{\rho(1-\rho^2)^{\frac{1}{2}}(\rho^2-s^2)^{\frac{1}{2}}} - \int_s^1 \frac{d\rho}{\rho(\rho^2-s^2)^{\frac{1}{2}}} \\ &= 2 \int_0^\phi \frac{1-\sin \theta}{\cos \theta} \frac{d\theta}{\{2(\cos 2\theta - \cos 2\phi)\}^{\frac{1}{2}}}, \quad \text{with } s = \cos \phi. \end{aligned} \quad (5.34)$$

Suppose then that

$$\frac{1-\sin \theta}{\cos \theta} = \sum_{m=0}^{\infty} c_m \cos (2m+1)\theta; \quad (5.35)$$

then

$$\frac{\sin^{-1} s}{s} = \frac{\pi}{2} \sum_{m=0}^{\infty} c_m P_m (2s^2-1). \quad (5.36)$$

The coefficients c_m are found to be given by

$$\frac{\pi}{4} c_m = (-1)^m \left\{ \frac{\pi}{2} - 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^m}{2m+1} \right) + \frac{(-1)^m}{2m+1} \right\}, \quad (5.37)$$

and we note that $c_m \sim (2m+1)^{-1}$ for large m . In a similar way we note that

$$(1-s^2)^{\frac{1}{2}} = \int_s^1 \frac{\rho d\rho}{(\rho^2-s^2)^{\frac{1}{2}}} = \int_0^\phi \frac{2 \sin \theta \cos \theta d\theta}{\{2(\cos 2\theta - \cos 2\phi)\}^{\frac{1}{2}}}. \quad (5.38)$$

Thus if

$$\frac{1}{2} \sin 2\theta = \sum_{m=0}^{\infty} d_m \cos (2m+1)\theta, \quad (5.39)$$

then

$$(1-s^2)^{\frac{1}{2}} = \frac{\pi}{2} \sum_{m=0}^{\infty} d_m P_m (2s^2-1). \quad (5.40)$$

The coefficients d_m are given by

$$\frac{\pi}{4} d_m = \frac{1}{2} \int_0^{\frac{1}{2}\pi} \sin 2\theta \cos (2m+1)\theta d\theta = -\frac{1}{(2m+3)(2m-1)}. \quad (5.41)$$

Thus

$$S(s) = H \sum_{m=0}^{\infty} e_m P_m - s^2, \quad e_m = c_m - d_m. \quad (5.42)$$

Since $S(s)/s^2$ is finite as $s \rightarrow 0$, we may rewrite (5.42) in the form

$$S(s) = H \sum_{m=0}^{\infty} f_m (P_m + P_{m+1}) - s^2, \quad (5.43)$$

where

$$f_m = e_m - e_{m-1} + \dots + (-1)^m e_0 \quad \text{and} \quad f_m \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty. \quad (5.44)$$

Then by using (5.27) we find that the coefficients g_n in (5.32) are given by

$$g_n = \frac{4H}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m f_{m+n}}{m+n+1} - \frac{2}{\pi} \delta_{n0}. \quad (5.45)$$

The equations (3.15) must be supplemented by the relation between the a_n and b_n . These may be found in three stages. The a_n are related to the α_n by (5.24), the b_n to the β_n by (5.30), and the α_n to the β_n by

$$\beta_n = \sum_{m=0}^{\infty} e_{m,n} \alpha_m, \quad (5.46)$$

where

$$\begin{aligned} e_{m,n} &= \frac{2}{\pi} \int_0^{\pi} \sin \theta \cos 2m\theta \cos 2n\theta d\theta \\ &= \frac{1}{\pi} \left(\frac{1}{2m+2n+1} - \frac{1}{2m+2n-1} + \frac{1}{2m-2n+1} - \frac{1}{2m-2n-1} \right). \end{aligned} \quad (5.47)$$

6. Case (iv)

The boundary conditions on T are

$$\frac{\partial T}{\partial \zeta} = 1, \quad \rho < 1; \quad \frac{\partial T}{\partial \zeta} - HT = 0, \quad \rho > 1. \quad (6.1)$$

We divide T into two parts:

$$T = T_1 + T_2, \quad (6.2)$$

where

$$\frac{\partial T_1}{\partial \zeta} = 1, \quad \rho < 1; \quad T_1 = 0, \quad \rho > 1. \quad (6.3)$$

The solution for T_1 is well known and, in particular

$$\left. \begin{aligned} T_1(\rho, 0) &= -\frac{2}{\rho} (1 - \rho^2)^{\frac{1}{2}}, \quad \rho < 1; \\ \frac{\partial T_1}{\partial \zeta}(\rho, 0) &= \frac{2}{\pi} \left\{ \sin^{-1} \left(\frac{1}{\rho} \right) - (\rho^2 - 1)^{-\frac{1}{2}} \right\}, \quad \rho > 1. \end{aligned} \right\} \quad (6.4)$$

Thus the boundary conditions for T_2 are

$$\frac{\partial T_2}{\partial \xi} = 0, \quad \rho < 1, \quad (6.5)$$

$$\frac{\partial T_2}{\partial \xi} - HT_2 = -\frac{2}{\pi} \left\{ \sin^{-1} \left(\frac{1}{\rho} \right) - (\rho^2 - 1)^{-\frac{1}{2}} \right\} \equiv R(\rho), \quad \rho > 1. \quad (6.6)$$

Again, since T_1 is continuous at $\rho = 1$, $z = 0$, and satisfies the zero-flux condition, T_2 will satisfy these conditions also.

Using the Hankel form (2.1) we find

$$\mathcal{H}_0[A(\xi); \rho] = 0, \quad \rho < 1, \quad (6.7)$$

$$\mathcal{H}_0[A(\xi); \rho] + H\mathcal{H}_0[\xi^{-1}A(\xi); \rho] = -R(\rho), \quad \rho > 1, \quad (6.8)$$

so that

$$\mathcal{F}_s[A(\xi); x] = (\tfrac{1}{2}\pi)^{\frac{1}{2}} f(x) U(x-1), \quad (6.9)$$

$$\frac{1}{\rho} \frac{d}{d\rho} \int_1^\rho \frac{xf(x) dx}{(\rho^2 - x^2)^{\frac{1}{2}}} + H \int_\rho^\infty \frac{f(x) dx}{(x^2 - \rho^2)^{\frac{1}{2}}} = -R(\rho), \quad \rho > 1. \quad (6.10)$$

Using the substitutions (5.14) we find

$$-s^2 \frac{d}{ds} s \int_s^1 \frac{j(y) dy}{(y^2 - s^2)^{\frac{1}{2}}} + H \int_0^s \frac{yj(y) dy}{(s^2 - y^2)^{\frac{1}{2}}} = -S(s), \quad (6.11)$$

where

$$S(s) = \frac{1}{s} R\left(\frac{1}{s}\right) = -\frac{2}{\pi} \left\{ \frac{\sin^{-1} s}{s} - (1 - s^2)^{-\frac{1}{2}} \right\}. \quad (6.12)$$

Put $y = \cos \theta$, $s = \cos \phi$ and use the representations

$$j(y) = J(\theta) = \sum_{n=1}^{\infty} a_n \sin 2n\theta, \quad (6.13)$$

$$\sin \theta J(\theta) = \sum_{n=1}^{\infty} \beta_n \sin 2n\theta. \quad (6.14)$$

The representation (6.14) automatically satisfies the zero-flux conditions.

Then

$$\begin{aligned} \int_s^1 \frac{j(y) dy}{(y^2 - s^2)^{\frac{1}{2}}} &= \sum_{n=1}^{\infty} \alpha_n \int_0^\phi \frac{2 \sin \theta \sin 2n\theta d\theta}{\{2(\cos 2\theta - \cos 2\phi)\}^{\frac{1}{2}}} \\ &= \frac{\pi}{4} \sum_{n=1}^{\infty} \alpha_n \{P_{n-1} - P_n\}, \end{aligned} \quad (6.15)$$

so that, on using (5.20), we find that

$$\begin{aligned} U_1 &\equiv -\frac{d}{ds} s \int_s^1 \frac{j(y) dy}{(y^2 - s^2)^{\frac{1}{2}}} = \frac{\pi}{4} \sum_{n=1}^{\infty} \alpha_n \{(2n-1)P_{n-1} + (2n+1)P_n\} \\ &= \frac{\pi}{4} \alpha_1 + \frac{\pi}{4} \sum_{n=1}^{\infty} (2n+1)(\alpha_n + \alpha_{n+1})P_n. \end{aligned} \quad (6.16)$$

Proceeding as in section 5 we note that

$$\frac{\sin (2n+1)\theta}{\cos \theta} = 2 \sin 2n\theta - 2 \sin (2n-2)\theta + \dots + 2(-1)^{n-1} \sin 2\theta + (-1)^n \frac{\sin \theta}{\cos \theta}, \quad (6.17)$$

so that if $J(\theta)$ is written in the form

$$\cos \theta J(\theta) = \sum_{n=0}^{\infty} a_n \sin (2n+1)\theta, \quad (6.18)$$

where

$$\sum_{n=0}^{\infty} (-1)^n a_n = 0, \quad (6.19)$$

then

$$\alpha_1 = 2a_0, \quad \alpha_n + \alpha_{n+1} = 2a_n, \quad (6.20)$$

so that

$$U_1 = \frac{\pi}{2} \sum_{n=0}^{\infty} (2n+1) a_n P_n. \quad (6.21)$$

The second integral can be treated similarly:

$$V_1 = \int_0^s \frac{y j(y) dy}{(s^2 - y^2)^{1/2}} = \frac{\pi}{4} \sum_{n=1}^{\infty} \beta_n \{P_n + P_{n-1}\}, \quad (6.22)$$

so that if (6.14) is rewritten as

$$\sin \theta J(\theta) = \sum_{n=0}^{\infty} (2n+1)^{-1} b_n \{n \sin 2n\theta + (n+1) \sin (2n+2)\theta\}, \quad (6.23)$$

where the b_n and β_n are related by equation (5.31), then

$$\frac{1}{s^2} V_1 = \frac{\pi}{2} \sum_{n=0}^{\infty} b_n P_n. \quad (6.24)$$

If therefore we expand $S(s)/s^2$ in the form (5.33) then equation (6.11) also reduces to equation (3.15).

The two remaining matters are the representation of $S(s)$ and the relation between the a_n and b_n . The first term in $S(s)$ is given in (5.36) and for the second we note that $(1-s^2)^{-1/2}$ has the formal expansion

$$(1-s^2)^{-1/2} = \frac{\pi}{2} \sum_{m=0}^{\infty} e_m P_m, \quad e_m = 4/\pi. \quad (6.25)$$

Now the g_n are found as in section 5. The a_n , α_n are related by (6.19), (6.20), the b_n , β_n by (5.31), and the α_n , β_n by

$$\beta_n = \sum_{m=1}^{\infty} f_{m,n} \alpha_m, \quad (6.26)$$

$$f_{m,n} = \frac{2}{\pi} \int_0^{\pi} \sin \theta \sin 2m\theta \sin 2n\theta d\theta$$

$$= \frac{1}{\pi} \left(\frac{1}{2m+2n-1} - \frac{1}{2m+2n+1} + \frac{1}{2m-2n+1} - \frac{1}{2m-2n-1} \right). \quad (6.27)$$

Finally, we note that the problem (6.6) may also be reduced to the Fredholm integral equation

$$f(x) + \frac{H}{\pi} \int_1^\infty f(t) \ln \left| \frac{(x^2-1)^{\frac{1}{2}} + (t^2-1)^{\frac{1}{2}}}{(x^2-1)^{\frac{1}{2}} - (t^2-1)^{\frac{1}{2}}} \right| dt = -\frac{2}{\pi} \int_1^x \frac{\rho R(\rho) d\rho}{(x^2-\rho^2)^{\frac{1}{2}}}, \quad x > 1. \quad (6.28)$$

This equation may be solved by approximation using (5.14), (6.13) and (6.14) or (6.18), (6.23).

7. Conclusions

Each of the four basic mixed boundary-value problems relating to the radiation condition has been reduced to an Abel type of integro-differential equation. In each case the use of appropriate Fourier representations reduces the problem basically to the same infinite set of simultaneous linear equations.

In cases (i) and (ii), it is known that the equations are regular. In cases (iii) and (iv), the regularity is not easily proved, but numerical experiments showed that the approximate solutions obtained by truncation exhibit good behaviour. Problems (ii) and (iii) are discussed further in the sequel.

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