Corrected copy.

SOME THERMOELASTIC CONTACT PROBLEMS INVOLVING FRICTIONAL HEATING

By J. R. BARBER

(*Department of Mechanical Engineering, University of Newcastle upon Tyne, NE1 7RU*)

[Received 14 April 1975]

SUMMARY

Solutions are obtained for some steady-state thermoelastic contact problems in which heat is generated due to friction at the interface between two semi-infinite solids. It is assumed that only one of the solids is a thermal conductor and that the shear tractions at the interface do not influence the normal tractions.

A general solution is obtained in terms of a single harmonic potential function which tends to zero on the surface outside the contact area, whilst inside this area a linear combination of the function and its normal derivative is prescribed. Approximate solutions are obtained for the particular cases in which the contact area is a circle or a strip and the surface of one of the contacting solids is spherical, cylindrical or plane.

1. Introduction

IF two semi-infinite elastic solids slide against each other, the generation of heat at the interface due to friction will induce temperature gradients and thermal stresses and there will generally be a consequent change in the distribution of contact stress and in the extent of the area of contact (**1**).

Elastic contact problems of this type are extremely intractable, since they involve moving sources of heat, combined normal and tangential loading of the solids and a contact area whose extent is not known *a priori*. However, some interesting analytical results can be obtained subject to the following simplifying assumptions:

(i) the contact area is stationary with respect to one solid, in which a steady flow of heat is established;
(ii) the other solid is a non-conductor;
(iii) the coupling between tangential and normal traction at the interface can be neglected;
(iv) the coefficient of friction is constant throughout the contact area.

Assumption (iii) is not required if the *elastic* properties of the two solids are similar.

In this paper, we shall reduce this simplified problem to that of determining a single harmonic potential function satisfying certain mixed boundary conditions. A general solution will be given for the axisymmetric case, making use of a method due to Collins (2).

The corresponding two-dimensional problem will also be briefly discussed in the interests of completeness. It has been previously treated by Burton (3, 4), but the present formulation provides a more rapidly convergent solution.

2. General solution

The heat input q_z to the thermally conducting solid must be equal to the rate of frictional heat generation throughout the contact area A. Hence

$$q_z = -\mu V \sigma_{zz} \quad \text{on } A, \tag{1a}$$

where μ is the coefficient of friction, V is the sliding speed and σ_{zz} is the normal contact stress. Tensile stresses are regarded as positive and the conducting solid occupies the space $z > 0$ in Cartesian coordinates (x, y, z) or cylindrical polar coordinates (r, θ, z). Outside the contact area, we assume that there is no heat flow from the surface, i.e.

$$q_z = 0 \quad \text{on } \bar{A}. \tag{1b}$$

The mechanical boundary conditions determining σ_{zz} can be stated as

$$u_z = u(x, y) \quad \text{on } A, \tag{2a}$$

$$\sigma_{zz} = 0 \quad \text{on } \bar{A}, \tag{2b}$$

$$\sigma_{xz} = \sigma_{yz} = 0 \quad \text{on } A \text{ and } \bar{A}, \tag{2c}$$

where u_z is the normal displacement at the surface and $u(x, y)$ is a prescribed function related to the shape of the contacting solids.

The statement (2c) that shear stress is zero even within the contact area deserves some comment in view of the fact that the same shear stresses are responsible for the heating effect under consideration. In fact, all that is assumed is that the shear stress has a negligible effect on the normal contact stress distribution, and hence that the boundary-value problem for determining the latter is the same as that which would be generated if the contact was frictionless, except for the inclusion of the appropriate heat input (1a). This assumption is exact if the two solids have similar mechanical properties since shear contact stresses will then produce equal and opposite normal displacements on the two surfaces and maintain conformity without changing the normal contact stress. The additional internal stresses and displacements due to frictional stresses can readily be calculated once the surface values are known.

A suitable general solution to the equations of steady-state thermo-elasticity in terms of two harmonic potential functions ϕ, ω is

$$\mathbf{u} = 4(1-\nu)\mathbf{k}\frac{\partial\phi}{\partial z} - 2(1-\nu)\nabla\phi + (1-2\nu)\nabla\omega + z\nabla\frac{\partial\omega}{\partial z} - (3-4\nu)\mathbf{k}\frac{\partial\omega}{\partial z}, \quad (3)$$

$$T = \frac{2(1-\nu)}{\alpha(1+\nu)}\frac{\partial^2\phi}{\partial z^2}, \quad (4)$$

where $\mathbf{u}$ is the displacement vector, T is the temperature rise above a suitable datum, and ν, α are respectively Poisson's ratio and the coefficient of thermal expansion for the material. This is solution B of reference (5). It is obtained from that given by Williams (6) by writing $\omega = \phi - \psi$.

The component of stress acting on the z-plane, $\mathbf{s}_z$, is given by

$$\frac{\mathbf{s}_z}{2G} = z\nabla\frac{\partial^2\omega}{\partial z^2} - \mathbf{k}\frac{\partial^2\omega}{\partial z^2}, \quad (5)$$

where G is the shear modulus, and the normal components of displacement and stress at the surface $z = 0$ reduce to

$$u_z = 2(1-\nu)\frac{\partial}{\partial z}(\phi - \omega), \quad (6)$$

$$\sigma_{zz} = -2G\frac{\partial^2\omega}{\partial z^2}. \quad (7)$$

The tangential stress at the surface is identically zero, as required by condition (2c), and the heat input per unit area is

$$q_z = -K\frac{\partial T}{\partial z} = -\frac{2K(1-\nu)}{\alpha(1+\nu)}\frac{\partial^3\phi}{\partial z^3}, \quad (8)$$

where K is the thermal conductivity of the material.

On examination of the boundary conditions (1b), (2b), it is clear that (1a) applies throughout $\bar{A}$ as well as A and hence this condition can be satisfied by taking

$$\frac{\partial\phi}{\partial z} \equiv -\beta\omega, \quad (9)$$

where
$$\beta = +\alpha\mu VG(1+\nu)/K(1-\nu). \quad (10)$$

Substituting for ϕ into equations (6) and (7) we obtain

$$u_z = -2(1-\nu)\left\{\beta\omega + \frac{\partial\omega}{\partial z}\right\}, \qquad z = 0, \quad (11)$$

$$\sigma_{zz} = -2G\frac{\partial^2\omega}{\partial z^2}, \qquad z = 0, \quad (12)$$

and the boundary-value problem is thereby reduced to the search for a single harmonic potential function ω in $z > 0$, satisfying

$$\beta\omega + \frac{\partial \omega}{\partial z} = -\frac{u(x, y)}{2(1-\nu)} \qquad \text{on } A, \tag{13a}$$

$$\frac{\partial^2 \omega}{\partial z^2} = 0 \qquad \text{on } \bar{A}. \tag{13b}$$

Since ω and its derivatives are harmonic, it follows that

$$\frac{\partial^2}{\partial z^2}\left\{\beta\omega + \frac{\partial \omega}{\partial z}\right\} = -\nabla_1^2\left\{\beta\omega + \frac{\partial \omega}{\partial z}\right\},$$

where $\nabla_1^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ and hence

$$\beta\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^3 \omega}{\partial z^3} = \frac{\nabla_1^2 u(x, y)}{2(1-\nu)} \qquad \text{on } A, \tag{13c}$$

from (13a).

Conditions (13b), (13c) define the boundary-value problem of the third kind for the harmonic function $\partial^2\omega/\partial z^2$. Similar boundary conditions are encountered in determining the steady-state temperature in a solid, part of whose boundary is maintained at zero temperature, whilst on the rest of the boundary there is radiation into a medium at a prescribed temperature. However, the sign of the constant, β, obtained in this example is opposite to that defined by equation (10) and the uniqueness theorem for the boundary-value problem of the third kind does not apply to the thermoelastic contact problem. This point will be clarified by the subsequent examples. We note that the methods used for solving this problem in sections 3.1, 4.2 below can equally be applied to the corresponding heat conduction problem, in which the radiating region is a circle and a strip respectively.

In solving the boundary-value problem defined by conditions (13b), (13c), certain continuity conditions must be imposed at the boundary between A and $\bar{A}$. If the surfaces of the contacting solids are continuous up to and including the first derivative, the contact stress and hence $\partial^2\omega/\partial z^2$ must tend to zero at this boundary. This condition is the same as that requiring continuity of potential (temperature) in the boundary-value problem of the third kind.

3. The axisymmetric problem

We assume that the contacting solids are initially convex, in which case the contact area must be simply connected† and hence a circle.

† It can be shown that the contact area in any thermoelastic contact problem is simply connected provided that the contacting solids are initially convex. The proof is derived from the argument of section 9 of reference (7).

Taking the radius, a, of this circle as the unit of length and defining non-dimensional coordinates $\rho = r/a$, $\zeta = z/a$, the problem is to find a harmonic function $V(\rho, \zeta)$ in $\zeta > 0$ satisfying

$$\frac{\partial V}{\partial \zeta} + a\beta V = f(\rho) \qquad \text{for} \quad 0 \leqslant \rho \leqslant 1, \quad z = 0, \tag{14a}$$

$$V = 0 \qquad \text{for} \quad \rho > 1, \quad z = 0, \tag{14b}$$

where

$$V \equiv \partial^2 \omega / \partial z^2 \tag{15}$$

and

$$f(\rho) = \frac{1}{2(1-\nu)\rho} \frac{d}{d\rho} \left\{ \rho \frac{d}{d\rho} \left(\frac{u(\rho)}{a} \right) \right\}. \tag{16}$$

This problem is treated by Collins (2), who gives the solution

$$V(\rho, \zeta) = \frac{1}{2i} \int_{-1}^{1} \frac{j(t)\, dt}{(\rho^2 + (\zeta + it)^2)^{\frac{1}{2}}}, \tag{17}$$

where $j(t)$ is an odd function of t satisfying the Fredholm integral equation

$$j(t) + \frac{a\beta}{\pi} \int_{-1}^{1} j(s) \log |s-t|\, ds = l(t), \qquad 0 \leqslant t \leqslant 1, \tag{18}$$

and

$$l(t) = \frac{2}{\pi t} \frac{d}{dt} \int_0^t \frac{\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \left\{ \int_0^\rho s f(s)\, ds \right\} d\rho. \tag{19}$$

Substituting for $f(\rho)$ from equation (16) and integrating, we obtain

$$l(t) = \frac{1}{\pi(1-\nu)t} \frac{d}{dt} \int_0^t \frac{\rho^2 \{d(u/a)/d\rho\} d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}. \tag{20}$$

3.1. The homogeneous equation

We first consider the case in which the surfaces are both plane, i.e. $u(\rho)$ is a constant. Substituting into equation (20), we find that $l(t)$ is identically zero and hence the Fredholm equation (18) is homogeneous.

To obtain an approximate solution, we write

$$s = \cos\theta, \qquad t = \cos\phi, \tag{21}$$

and represent $j(\phi)$ in the form

$$j(\phi) = \sum_{i=1}^{\infty} a_i \cos(2i-1)\phi, \qquad 0 \leqslant \phi \leqslant \pi, \tag{22}$$

giving

$$\sum_{i=1}^{\infty} a_i \cos(2i-1)\phi + \\ + \frac{a\beta}{\pi} \int_0^\pi \sum_{j=1}^{\infty} a_j \cos(2j-1)\theta \log|\cos\theta - \cos\phi| \sin\theta\, d\theta = 0, \\ 0 \leqslant \phi \leqslant \pi. \tag{23}$$

Only odd multiples of ϕ are required in the series (22) since $j(\phi)$ is odd with respect to $\frac{1}{2}\pi$.

The expression $\sin\theta\cos(2j-1)\theta$ can be expanded as a Fourier series in the range $0 \leqslant \theta \leqslant \pi$ as

$$\sin\theta\cos(2j-1)\theta = \sum_{i=1}^{\infty} b_{ij}\cos(2i-1)\theta, \tag{24}$$

where $$b_{ij} = -\frac{2}{\pi}\left\{\frac{1}{4(i-j)^2-1}+\frac{1}{4(i+j-1)^2-1}\right\}. \tag{25}$$

We now substitute into equation (23) and perform the integration, making use of the result

$$\int_0^{\pi}\cos n\theta \log|\cos\theta-\cos\phi|\,d\theta = -\frac{\pi\cos n\phi}{n}, \qquad n \neq 0, \tag{26}$$

Note.
$n=0$ gives $-\pi\log 2$

(see (8)) to obtain

$$\sum_{i=1}^{\infty} a_i\cos(2i-1)\phi - a\beta\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{b_{ij}a_j\cos(2i-1)\phi}{(2i-1)} = 0 \tag{27}$$

and hence, equating coefficients of $\cos(2i-1)\phi$,

$$\sum_{j=1}^{\infty} b_{ij}a_j = (2i-1)a_i/a\beta, \qquad i = 1, 2, 3, \ldots. \tag{28}$$

The matrix of coefficients of this system of equations can be made symmetric by the trivial change of variable $x_i = a_i(2i-1)^{\frac{1}{2}}$. We can therefore deduce that all the eigenvalues are real. Furthermore, they must all be positive, since negative eigenvalues are ruled out by the uniqueness theorem for the boundary-value problem of the third kind (9).

On truncating the series to twelve terms, the first six eigenvalues of $\pi/2a\beta$ are found as 0·7830, 0·3065, 0·1902, 0·1378, 0·1080 and 0·0889. The coefficient matrix is strongly diagonal and gives a rapid convergence with increasing number of terms. For example, a change from ten to twelve terms only affects the first five eigenvalues in the sixth significant figure. An alternative iterative solution for the first eigenvalue showed no further change in the sixth significant figure from the value 0·783002 for increases in the number of terms beyond eleven.

To find the corresponding contact stress distribution, we substitute into equation (17), noting that when $\zeta = 0$,

$$V(\rho, 0) = -\int_{\rho}^{1}\frac{j(t)\,dt}{(t^2-\rho^2)^{\frac{1}{2}}}, \qquad 0 \leqslant \rho \leqslant 1. \tag{29}$$

The contact stress distributions corresponding to the first three eigenvalues are shown graphically in Fig. 1. Only the first eigenvalue satisfies the requirement that contact stress should be everywhere compressive. In

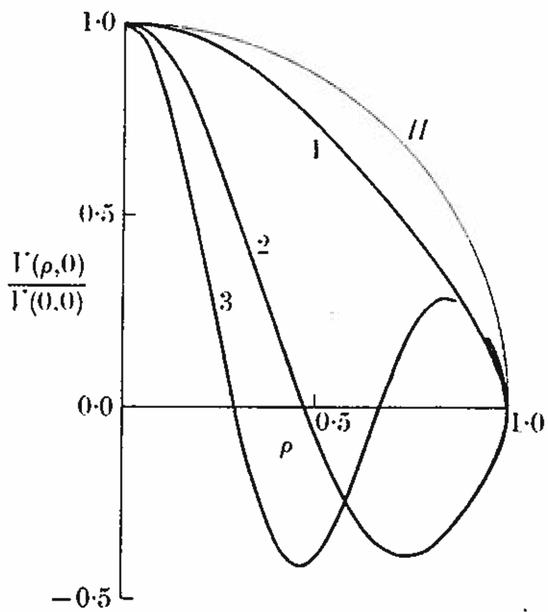

FIG. 1. Contact pressure distributions associated with the first 3 eigenvalues of $\pi/2a\beta$. The Hertzian pressure distribution H is given for comparison.

general, the contact stress for the nth eigenvalue changes sign $n-1$ times in the interval $0 < \rho \leqslant 1$. Thus, the condition that contact stress should be non-tensile guarantees uniqueness for this particular contact geometry.

A previous estimate for the first eigenvalue was obtained by Burton (**10**) using a different method in which the contact stress (and hence also the heat input) was represented by a truncated power series in ρ. A corresponding power series was obtained for the displacement due to combined heating and loading, and a system of simultaneous equations was thereby obtained for the coefficients. This method works quite well for the two-dimensional case (see below, section 4), but the mathematical complexity of the axisymmetric case persuaded Burton to restrict his series to only three terms and hence the value he obtained (0·84) does not agree very closely with that derived from the present more exact method. We might also note that a power series in ρ is not well adapted to the representation of a function which tends to zero with $(1-\rho^2)^{\frac{1}{2}}$ as $\rho \to 1$.

3.2. Contact of a sphere and a plane

We now consider the non-homogeneous case in which the surface of the conducting solid is slightly spherical. If the radius of the sphere is R, the function $u(\rho)$ takes the form

$$u(\rho) = -\frac{a^2\rho^2}{2R} + \text{constant}, \tag{30}$$

and the function $l(t)$ is obtained from equation (20) as

$$l(t) = -\frac{2at}{\pi(1-\nu)R} = -\frac{2a\cos\phi}{\pi(1-\nu)R}. \tag{31}$$

Thus, the integral equation (18) takes the form

$$j(\phi)+\frac{a\beta}{\pi}\int_0^\pi j(\theta)\log|\cos\theta-\cos\phi|\sin\theta\,d\theta = -\frac{2a\cos\phi}{\pi(1-\nu)R},$$

$$0 \leqslant \phi \leqslant \pi, \tag{32}$$

and using the same procedure as in section 3.1, this can be expressed as

$$\sum_{i=1}^{\infty} a_i\cos(2i-1)\phi - a\beta\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{b_{ij}a_j\cos(2i-1)\phi}{(2i-1)} = -\frac{2a\cos\phi}{\pi(1-\nu)R}. \tag{33}$$

Writing

$$x_i = \pi(1-\nu)Ra_i/2a, \tag{34}$$

and equating coefficients, we have

$$a\beta\sum_{j=1}^{\infty} b_{ij}x_j-(2i-1)x_i = \delta_{i1} \quad \text{for} \quad i = 1, 2, 3, \ldots, \tag{35}$$

where the b_{ij} are given by equation (25) and δ_{ij} is the Kronecker delta.

3.3. Relation between load and contact area

It is of interest to discover how the radius of the contact area a varies with applied load P, sliding speed V and the radius of the sphere R. As in the isothermal Hertzian contact problem, it is convenient to find the applied load, treating the contact radius as known, and then invert the relationship.

The load P is given by

$$P = -\int_0^a 2\pi r\sigma_{zz}(r,0)\,dr = 4\pi a^2G\int_0^1 \rho V(\rho,0)\,d\rho, \tag{36}$$

from equations (12) and (15).

Substituting for $V(\rho, 0)$ from equation (29), we obtain

$$P = -4\pi a^2G\int_0^1\int_\rho^1 \frac{\rho j(t)\,dt\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}. \tag{37}$$

If we reverse the order of integration and perform the inner integral, this reduces to

$$P = -4\pi a^2G\int_0^1 tj(t)\,dt. \tag{38}$$

Finally, substituting for $j(t)$ from equations (21) and (22) we obtain

$$P = -4\pi a^2 G \int_0^{\frac{1}{2}\pi} \sum_{i=1}^{\infty} a_i \cos\phi \cos(2i-1)\phi \sin\phi \, d\phi$$

$$= \frac{8a^3G}{(1-\nu)R} \sum_{i=1}^{\infty} \frac{x_i}{(2i+1)(2i-3)}. \tag{39}$$

For the trivial case $\beta = 0$, we have the Hertzian solution

$$P = P_H = \frac{8a^2G}{3(1-\nu)R}, \tag{40}$$

and hence equation (39) can be expressed in the non-dimensional form

$$P/P_H = 3 \sum_{i=1}^{\infty} \frac{x_i}{(2i+1)(2i-3)}. \tag{41}$$

This expression is a function of $a\beta$ only (see equation (35)) which increases without limit as $a\beta$ approaches the first eigenvalue. The reciprocal P_H/P obtained using a series of twenty terms is plotted against $a\beta$ in Fig. 2.

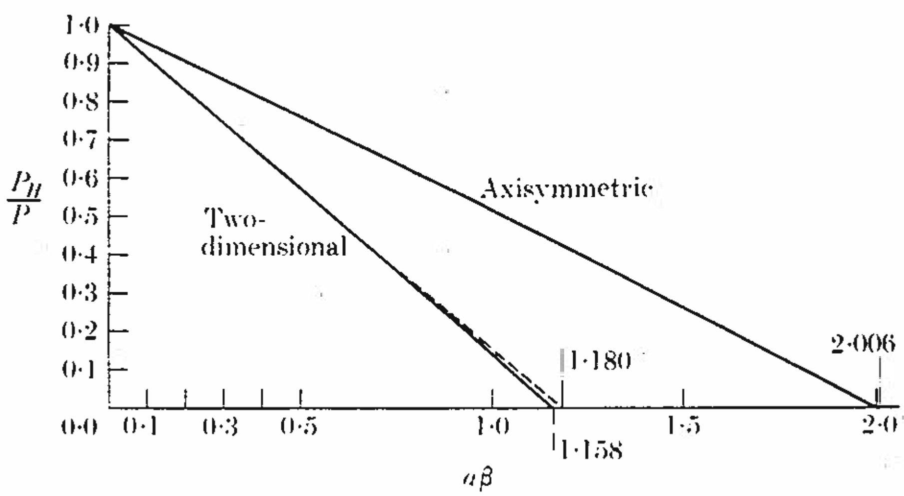

FIG. 2. Relationship between the reciprocal of non-dimensional load and contact area for the axisymmetric and two-dimensional geometries.

The solution of equation (35) only satisfies the requirement of non-tensile contact stress in the range $0 \leqslant a\beta \leqslant 2{\cdot}006$.

The thermoelastic contact area is always smaller than the Hertzian, the difference being most pronounced at high speeds and loads when the radius approaches the plane surface value

$$a = \frac{2{\cdot}006K(1-\nu)}{\alpha\mu VG(1+\nu)}. \tag{42}$$

If the load is increased at constant speed, this expression defines a constant limit which the contact radius approaches but cannot exceed. (In the corresponding isothermal case the contact radius increases with load without limit.)

If the load is kept constant whilst the speed is increased, the contact radius is reduced without limit, approaching inverse proportionality with speed as defined by equation (42) at very high speeds.

4. The two-dimensional problem

The corresponding two-dimensional (plane strain) problem is that in which two semi-infinite solids make contact over the strip $-a \leqslant x \leqslant a$, the geometry and contact pressure being independent of y. Writing $\xi = x/a$, $\zeta = z/a$, the problem is to find a two-dimensional harmonic function $V(\xi, \zeta)$ in $\zeta > 0$ satisfying

$$\frac{\partial V}{\partial \zeta}+a\beta V = f(\xi), \qquad |\xi| \leqslant 1, \quad z = 0, \tag{43a}$$

$$V = 0, \qquad |\xi| > 1, \quad z = 0, \tag{43b}$$

where

$$f(\xi) = \frac{1}{2(1-\nu)}\frac{d^2}{d\xi^2}\left\{\frac{u}{a}\right\}, \tag{44}$$

(cf. equations (14) and (15)).

The problem can be reduced to an integral equation, using an argument similar to that of Collins (2), but the kernel of the resulting equation is a complete elliptic integral which leads to a complicated approximate solution. In fact, it proves more straightforward to follow Burton (3, 4) in representing the unknown potential V as a series in the contact region and developing a corresponding integral expression for $\partial V/\partial \zeta$. However, the use of a Fourier series in preference to the power series used by Burton conduces to much faster convergence (see below).

Assuming $V(\xi, 0)$ known in the contact region, we have

$$\frac{\partial V}{\partial \zeta}(\xi, 0) = \frac{1}{\pi}\frac{d^2}{d\xi^2}\int_{-1}^{1} V(s, 0)\log|s-\xi|\,ds. \tag{45}$$

The function $V(s, 0)$ must be an even function of s tending to zero at $s = \pm 1$ and we can represent it as

$$V(\theta, 0) = \sum_{i=1}^{\infty} a_i \sin (2i-1)\theta, \qquad 0 \leqslant \theta \leqslant \pi, \tag{46}$$

where $s = \cos \theta$, $\xi = \cos \phi$.

Substituting into equation (45, 43a) and making use of equation (26), we find

$$a\beta \sum_{j=1}^{\infty} a_j \sin (2j-1)\phi - \sum_{i=1}^{\infty} \frac{(2i-1)a_i \sin (2i-1)\phi}{\sin \phi} = f(\phi), \qquad 0 \leqslant \phi \leqslant \pi. \tag{47}$$

We now multiply through by $\sin \phi$ and expand $\sin \phi \sin (2i-1)\phi$, as in section 3.1, to obtain

$$a\beta \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} a_j \sin (2i-1)\phi - \sum_{i=1}^{\infty} (2i-1)a_i \sin (2i-1)\phi = f(\phi) \sin \phi, \qquad 0 \leqslant \phi \leqslant \pi, \tag{48}$$

where
$$c_{ij} = -\frac{2}{\pi}\left\{\frac{1}{4(i-j)^2-1} - \frac{1}{4(i+j-1)^2-1}\right\}. \tag{49}$$

4.1. The homogeneous equation

On truncating the series to twelve terms and setting $f(\phi) \equiv 0$, the first six eigenvalues of $\pi/2a\beta$ from equation (48) are found as 1·3567, 0·3639, 0·2106,

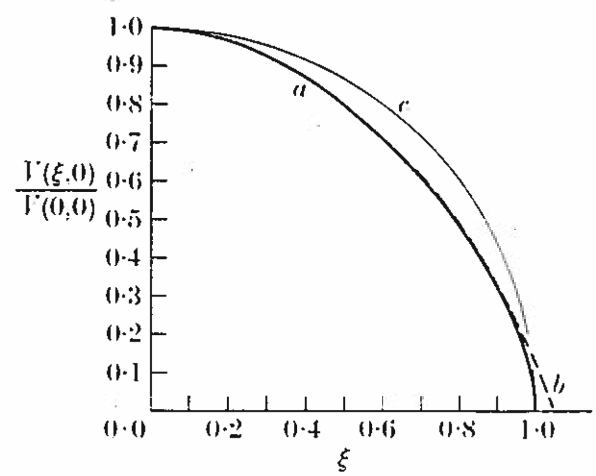

FIG. 3. Contact pressure distribution in the two-dimensional thermoelastic contact of plane surfaces (a), compared with Burton's power series approximation (b), and the Hertzian pressure distribution (c).

0·1482, 0·1143 and 0·0930. The diagonality of the matrix c_{ij} is even more pronounced than in the axisymmetric case. If all the non-diagonal elements are neglected, the value $\frac{4}{3}$ obtained for the first eigenvalue only differs from the more accurate estimate by 2%.

The first eigenvalues show no further change in the sixth significant figure from the value of 1·35674 for increase in the number of terms beyond four. By contrast, Burton's power series representation gives a value which still changes in the third figure after twenty terms (3). His estimate for $\pi/2a\beta$, based on an extrapolation through the trend with increasing numbers of terms is 1·362.

As before, the first eigenvalue is the only one which is consistent with the requirement of non-tensile contact stress. The contact stress distribution is shown in Fig. 3 in comparison with the isothermal Hertzian solution and with Burton's power-series approximation.

4.2. Contact of a cylinder on a plane

We now consider the case in which the surface of the conducting solid is cylindrical with radius R, for which

$$u(\xi) = -\frac{a^2\xi^2}{2R} + \text{constant} \tag{50}$$

and hence

$$f(\phi) = -a/2R(1-\nu), \tag{51}$$

which is independent of ϕ.

Substituting into equation (48) and equating coefficients, we find

$$a\beta \sum_{j=1}^{\infty} c_{ij} x_j - (2i-1)\, x_i = -\delta_{i1} \quad \text{for} \quad i = 1, 2, 3, \ldots, \tag{52}$$

where

$$x_j = 2R(1-\nu)a_j/a. \tag{53}$$

The total applied load is

$$\begin{aligned} P &= 2Ga \int_{-1}^{1} V(\xi, 0)\, d\xi \\ &= \frac{Ga^2}{R(1-\nu)} \int_0^{\pi} \sum_{i=1}^{\infty} x_i \sin\phi \sin(2i-1)\phi\, d\phi \\ &= \frac{\pi G a^2 x_1}{2R(1-\nu)}. \end{aligned} \tag{54}$$

If $\beta = 0$ (the isothermal Hertzian case), we have $x_1 = 1$, from equation (52), and hence

$$P/P_H = x_1. \tag{55}$$

Values of the reciprocal of this ratio, obtained using a series of twenty terms, are shown graphically in Fig. 2. The effect of load and speed on contact width is qualitatively similar to that in the axisymmetric case.

If the series is truncated to one term, we obtain

$$\frac{P_H}{P} = 1 - \frac{8\alpha\beta}{3\pi}. \tag{56}$$

This result is shown dotted in Fig. 2 and clearly provides a very acceptable approximation to the more exact result.

REFERENCES

1. J. R. Barber, *Proc. R. Soc.* **A312** (1969) 381.
2. W. D. Collins, *Mathematika* **6** (1959) 120.
3. R. A. Burton, S. R. Kilaparti and V. Nerlikar, *Wear* **24** (1973) 199.
4. R. A. Burton and V. Nerlikar, *ibid.* **27** (1974) 195.
5. J. R. Barber, *Proc. IUTAM Symposium on the Mechanics of the Contact between Deformable Bodies*, Twente (1974).
6. W. E. Williams, *Z. angew. Math. Phys.* **12** (1961) 452.
7. J. R. Barber, *Int. J. Heat Mass Transfer* **14** (1971) 751.
8. Yu. V. Pimenov, *Soviet Phys. Tech. Phys.* **4** (1960) 532.
9. W. J. Sternberg and T. L. Smith, *The Theory of Potential and Spherical Harmonics* (University of Toronto Press, 1944) 294.
10. R. A. Burton, *Proc. IUTAM Symposium on the Mechanics of the Contact between Deformable Bodies*, Twente (1974).