Incremental stiffness and electrical contact conductance in the contact of rough finite bodies

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If two half spaces are in contact, there exists a formal mathematical relation between the electrical contact resistance and the incremental elastic compliance. Here, this relation is extended to the contact of finite bodies. In particular, it is shown that the additional resistance due to roughness of the contacting surfaces (the interface resistance) bears a similar relation to the additional compliance as that obtained for the total resistance in the half-space problem.

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I. INTRODUCTION

The effect of surface roughness on electrical and thermal contact resistance has been a subject of scientific interest since the early days of tribology. Indeed the experimental observation that electrical contact conductance (reciprocal of resistance) is approximately proportional to the normal contact force (in contrast to that for a single smooth Hertzian contact) was a significant motivation for the derivation of the classical contact theories of Greenwood and Williamson [1] and Archard [2]. Greenwood and Williamson's theory predicts that the principal effect of increasing the normal force is to increase the *number* of asperity contacts, with the distribution of states remaining relatively unchanged. It then follows that any physical quantities that can be written as a sum over the set of contacting asperities should be approximately linearly proportional to each other. Greenwood and Williamson's theory neglects the interaction between adjacent asperities, which is a serious limitation for the contact of fractal surfaces, where contacts tend to be clustered [3]. However, a similar linear relation is obtained from Persson's theory of rough surface contact [4,5], where it follows from the observation that the probability $\mathcal{P}(p)$ of a given point on the surface being in contact at pressure p tends to a limiting form as the contributions of the finer scales of the power spectral density (PSD) of the surface are included [6].

Numerous experimental studies have been conducted into the effect of contact force on thermal contact conductance, and the results are usually approximated by power-law relations, with exponents that are often somewhat lower than linearity [7,8]. If the contact is elastic, it can be shown that the contact resistance is proportional to the incremental elastic compliance [9], and many numerical studies of this latter problem have been conducted, some appearing to confirm a linear relation [10,11], while others suggest a power-law behavior, with the exponent depending on the fractal dimension or Hurst exponent of the roughness [12,13].

Persson *et al.* [14] have recently suggested that this discrepancy might be attributable to finite size effects. It is generally accepted that contact resistance is predominantly determined at the coarse scale of the roughness spectrum [9], and both numerical and experimental models must of necessity be of finite size in all dimensions. This implies that there must be a longest wavelength in the roughness, and in practical systems there will generally also be a deviation from fractality of the PSD in this range which might be crucial to the predictions. Furthermore, the finite dimensions of the bodies will directly affect the overall resistance, and it is not a trivial matter to distinguish the contribution of the surface roughness from that of the underlying structure.

In this paper, we shall pay particular attention to the effect of the finite size of the body on the relation between contact resistance and incremental elastic stiffness. Sevostianov [15] has shown that the relation established in Ref. [9] and given in Eq. (1) below applies to the *additional* resistance and compliance associated with an axisymmetric region of noncontact in the case of two smooth cylinders in contact on their end faces. Here we seek to establish a more general result, subject only to the restriction that the longest wavelength in the surface roughness is significantly smaller than the macroscopic dimensions of the system.

II. RELATION BETWEEN CONTACT RESISTANCE AND INCREMENTAL STIFFNESS

If the elastic and resistive bodies can reasonably be approximated by half spaces, the boundary-value problems for electrical conduction and elastic indentation can be expressed in mathematically identical form, and this enables us to prove rigorously that the electrical resistance R is given by Ref. [9]

$$\frac{1}{R} = \frac{2}{E^*(\rho_1+\rho_2)}\frac{dF}{dw}, \tag{1}$$

where

$$\frac{1}{E^*} = \frac{\left(1-\nu_1^2\right)}{E_1} + \frac{\left(1-\nu_2^2\right)}{E_2} \tag{2}$$

is the composite elastic modulus, E_i, ν_i, and ρ_i ($i = 1,2$) are Young's modulus, Poisson's ratio, and electrical resistivity for the two contacting bodies, respectively, F is the applied normal contact force, and w is the relative normal displacement of distant regions of the two bodies.

In finite geometry problems, the half-space approximation is reasonable in an asymptotic sense, provided that the contact region is small compared with the other dimensions of the body, since the perturbations in both electrical potential and elastic displacement decay with the reciprocal of distance from the contact region. In fact, the same approximation is

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made in the classical Hertzian theory of elastic contact [16] and in Holm's theory of electrical contact resistance [17]. This permits Eq. (1) to be applied to a fairly general class of problems, including, notably, cases where the contacting surfaces are rough.

However, in more general finite geometry problems, Eq. (1) does not generally apply. For example, if a bar of length L and cross-sectional area A makes perfect electrical frictionless contact with rigid planes at its two ends, the electrical resistance R and the incremental stiffness dF/dw are easily obtained as

$$R = \frac{\rho L}{A}, \quad \frac{dF}{dw} = \frac{EA}{L}.$$

It follows that if two such blocks of lengths L_1,L_2 and material properties ρ_1,ρ_2,E_1,E_2 are pressed together, the combined resistance and incremental compliance are

$$R = \frac{(\rho_1 L_1 + \rho_2 L_2)}{A}, \quad \frac{dw}{dF} = \frac{1}{A}\left(\frac{L_1}{E_1} + \frac{L_2}{E_2}\right),$$

which clearly do not permit a more general relation of the form (1).

It should be emphasized that R and dF/dw in Eq. (1) refer to the "total" electrical resistance and incremental compliance, respectively. In other words, these quantities are defined in terms of the values of electrical potential and elastic displacement "at infinity" in the contacting bodies. Thus, for example, in the case of a sphere indenting a rough half space, the contact resistance will include both the "constriction resistance" due to the surface roughness and an additional component, sometimes called the "cluster resistance," due to the fact that all the areas of actual contact are grouped in a region comparable with that which would be established between corresponding smooth bodies [18,19].

More generally, it is often assumed that surface roughness interposes a kind of resistive and compliant layer between the contacting bodies, such that the additional compliance $w_{\mathcal{I}}$ is some nonlinear function of the nominal contact pressure p. We shall refer to $w_{\mathcal{I}}$ as the *interface compliance* and the corresponding additional contact resistance as the *interface resistance* $R_{\mathcal{I}}$. Notice that these quantities must be defined "per unit area," so that, for example, $R_{\mathcal{I}}$ will be the additional potential drop across the interface due to roughness, divided by the nominal current density $\bar{\imath}$ [i.e., not by the total current as in Eq. (1)]. It can also be useful to define the thickness of a layer of material whose resistance and/or compliance are equal to the interface resistance and compliance, respectively [20]. If the properties of this fictitious layer can be established, they can be used to define an appropriate boundary condition in, e.g., a numerical solution of more general finite geometry elastic or electrical contact problems involving rough surfaces.

The interface resistance can be expected to depend on p but be independent of the actual size of the contacting bodies, provided the system exhibits *scale separation*, i.e., that the longest wavelength in the roughness spectrum is significantly smaller than the finite dimensions of the system. Indeed, if this condition is *not* satisfied, the statistical variance in the roughness description is likely to impose an unacceptable level of variance in the total resistance and compliance of the system in the sense that different realizations of the same statistics could exhibit significantly different macroscopic behavior.

Under these conditions (of scale separation) it seems intuitively reasonable that the incremental interface stiffness $dp/dw_{\mathcal{I}}$ would be related to the interface resistance $R_{\mathcal{I}}$ in the same way as the total incremental stiffness and resistance through Eq. (1) since, if we focus our attention on a small region comparable to the scale of the microscopic roughness, the bodies will appear asymptotically to be semi-infinite. However, the half-space problem so generated involves contact throughout the interfacial plane and hence violates the condition that contact be restricted to a region that is small compared with the dimensions of the body. Furthermore, the total resistance and total incremental compliance [which appear in Eq. (1)] cease to be meaningful concepts in this limit since, for example, if a uniform pressure is applied over the entire surface of a half space, the compliance will be infinite. We shall show that an equation similar to Eq. (1) *does* relate $dp/dw_{\mathcal{I}}$ and $R_{\mathcal{I}}$, but we need to use a more careful asymptotic procedure to establish it rigorously.

III. MATHEMATICAL PRELIMINARIES

An essential tool in the establishment of the required relationship is the representation of elastic fields in terms of a harmonic potential function φ developed by Green and Zerna [21,22]. The corresponding components of stress and displacement are given by

$$2\mu u_x = z\frac{\partial^2\varphi}{\partial x\partial z} + (1-2\nu)\frac{\partial\varphi}{\partial x}, \tag{3}$$

$$2\mu u_y = z\frac{\partial^2\varphi}{\partial y\partial z} + (1-2\nu)\frac{\partial\varphi}{\partial y}, \tag{4}$$

$$2\mu u_z = z\frac{\partial^2\varphi}{\partial z^2} - 2(1-\nu)\frac{\partial\varphi}{\partial z}, \tag{5}$$

$$\sigma_{xx} = z\frac{\partial^3\varphi}{\partial x^2\partial z} + \frac{\partial^2\varphi}{\partial x^2} + 2\nu\frac{\partial^2\varphi}{\partial y^2}, \tag{6}$$

$$\sigma_{xy} = z\frac{\partial^3\varphi}{\partial x\partial y\partial z} + (1-2\nu)\frac{\partial^2\varphi}{\partial x\partial y}, \tag{7}$$

$$\sigma_{yy} = z\frac{\partial^3\varphi}{\partial y^2\partial z} + \frac{\partial^2\varphi}{\partial y^2} + 2\nu\frac{\partial^2\varphi}{\partial x^2}, \tag{8}$$

$$\sigma_{zx} = z\frac{\partial^3\varphi}{\partial z^2\partial x}, \quad \sigma_{zy} = z\frac{\partial^3\varphi}{\partial z^2\partial y}, \tag{9}$$

$$\sigma_{zz} = z\frac{\partial^3\varphi}{\partial z^3} - \frac{\partial^2\varphi}{\partial z^2}, \tag{10}$$

where μ is the shear modulus of the material. Notice in particular that the stress components σ_{zx},σ_{zy} are both zero throughout the plane $z=0$ for all functions φ, so that this solution is particularly useful for problems in which frictionless contact occurs on that plane.

In the context of the half-space problem, the incremental elastic contact problem is defined by the boundary conditions $u_z(x,y,0) = \Delta w$ in the contact region $\mathcal{A}$ and $\sigma_{zz}(x,y,0) = 0$ (traction free) in the separation region $\bar{\mathcal{A}}$, where Δw is the incremental normal compliance. If the above solution is used to describe the elastic fields, this defines the boundary-value problem

$$\frac{\partial\varphi}{\partial z}(\mathcal{A}) = -\frac{\mu\Delta w}{(1-\nu)}, \quad \frac{\partial^2\varphi}{\partial z^2}(\bar{\mathcal{A}}) = 0. \tag{11}$$

The corresponding conduction problem for the electrical potential V is defined by the boundary conditions

$$V(\mathcal{A}) = V_0, \quad \frac{\partial V}{\partial z}(\bar{\mathcal{A}}) = 0, \tag{12}$$

and a comparison of these two sets of conditions clearly exposes the mathematical similarity between the harmonic potentials V and $\partial\varphi/\partial z$, leading to Eq. (1) [9].

IV. THE RECTANGULAR BLOCK

To explore the effect of finite geometry, we consider the system illustrated in Fig. 1, in which the rectangular block $-a<x<a, -a<y<a, 0<z<b$ rests on a frictionless rigid plane at $z=b$. The upper surface $z=0$ is then loaded by frictionless contact with a rough plane surface, such that areas of actual contact are established as shown. The remaining surfaces are traction free. The contacting rough surface is assumed to be a rigid perfect conductor that is maintained at electrical potential V_0, the surface $z=b$ is maintained at zero potential, and there is no current flow at the remaining surfaces, so, for example,

$$\rho i_x = -\frac{\partial V}{\partial x} = 0 \quad \text{on} \quad x = \pm a. \tag{13}$$

The electrical boundary conditions in the actual contact area $\mathcal{A}$ and the separation region $\bar{\mathcal{A}}$ are

$$V = V_0 \quad \text{in} \quad \mathcal{A}, \tag{14}$$

$$\frac{\partial V}{\partial z} = 0 \quad \text{in} \quad \bar{\mathcal{A}}. \tag{15}$$

If the contact were perfect, i.e., if $\mathcal{A}$ comprised the whole surface $z=0$ of the block, the current flow would be uniform and unidirectional ($i_z = i$), the potential difference V_0 and the current density i would be related by $V_0 = \rho i b$, and the potential field would be defined by $V(x,y,z) = \rho i(b-z)$. When the contact is imperfect, it is convenient to define a new potential V_1 such that

$$V(x,y,z) = V_1(x,y,z) + \rho\bar{\imath}(b-z) \tag{16}$$

or

$$V_1(x,y,z) = V(x,y,z) - \rho\bar{\imath}(b-z), \tag{17}$$

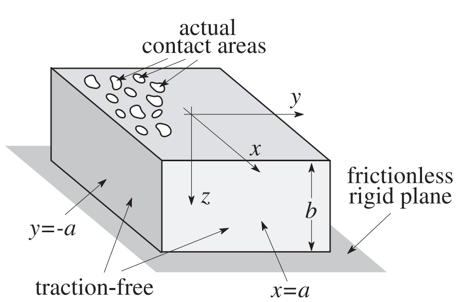

FIG. 1. The rectangular block.

where

$$\bar{\imath} = \frac{1}{4a^2}\int_{-a}^{a}\int_{-a}^{a} i_z(x,y,z)dxdy \tag{18}$$

is now the nominal (average) current density. We also define the interface resistance $R_{\mathcal{I}}$, such that

$$V_0 = \bar{\imath}R_{\mathcal{I}} + \rho\bar{\imath}b. \tag{19}$$

With this definition, it is clear that $\bar{\imath}R_{\mathcal{I}}$ represents the value of $V_1(x,y,z)$ in $\mathcal{A}$.

We suppose that the maximum wavelength of the roughness $\lambda \ll a,b$ so that the nonuniformity of the potential field has decayed before we reach the back surface. Thus, in this region ($z \gg \lambda$), the potential has the asymptotic form

$$V(x,y,z) \to \rho\bar{\imath}(b-z), \quad \frac{\partial V}{\partial z} \to -\rho\bar{\imath}, \tag{20}$$

and it follows that

$$V_1(x,y,z) \to 0, \quad \frac{\partial V_1}{\partial z} \to 0, \quad z \gg \lambda. \tag{21}$$

A. The elastic problem

Suppose that the electrical problem has been completely solved, so that the potential $V(x,y,z)$ is a known function. We shall develop the solution to the elasticity problem in two steps by (a) applying the potential

$$\frac{\partial\varphi}{\partial z} = -\frac{pV(x,y,z)}{\rho\bar{\imath}} \tag{22}$$

in the solution of Eqs. (3)–(10), followed by (b) a correction to satisfy the traction-free condition on $x=\pm a$, $y=\pm a$.

We first note that in the contact area $\mathcal{A}$, we have

$$2\mu u_z(\mathcal{A}) = -2(1-\nu)\frac{\partial\varphi}{\partial z} = \frac{2(1-\nu)pV_0}{\rho\bar{\imath}} \tag{23}$$

from Eqs. (5), (14), and (22), or in terms of $R_{\mathcal{I}}$,

$$2\mu u_z(\mathcal{A}) = \frac{2(1-\nu)pR_{\mathcal{I}}}{\rho} + 2(1-\nu)pb. \tag{24}$$

When $z \gg \lambda$ and, in particular, near $z=b$, we have

$$\frac{\partial^2\varphi}{\partial z^2} = -\frac{p}{\rho\bar{\imath}}\frac{\partial V}{\partial z} \to p, \quad \frac{\partial^3\varphi}{\partial z^3} \to 0, \tag{25}$$

so

$$\sigma_{zz} = -p, \quad \sigma_{zx} = \sigma_{zy} = 0, \tag{26}$$

as required. Also, in the same region

$$2\mu u_z = z\frac{\partial^2\varphi}{\partial z^2} - 2(1-\nu)\frac{\partial\varphi}{\partial z} \to pz + 2(1-\nu)p(b-z), \tag{27}$$

so on the surface $z=b$, we have

$$2\mu u_z \to pb. \tag{28}$$

Equations (26) and (28) satisfy the conditions for frictionless contact with a rigid plane at $z=b$ with uniform contact pressure p (which is also the nominal contact pressure on the plane $z = 0$). It then follows that the contact compliance in solution (a) is

$$w_{(a)} = u_z(0) - u_z(b) = \frac{(1-\nu)pR_{\mathcal{I}}}{\mu\rho} + \frac{(1-\nu)pb}{\mu} - \frac{pb}{2\mu}$$
$$= \frac{2(1-\nu^2)pR_{\mathcal{I}}}{E\rho} + \frac{(1-2\nu)(1+\nu)pb}{E}, \tag{29}$$

using the relation $\mu = E/2(1+\nu)$.

1. The lateral surfaces

In order to calculate the full stress and displacement fields associated with solution (a), we need to perform a partial integration on Eq. (22) with respect to z. We define

$$\varphi = \int \frac{\partial\varphi}{\partial z}dz + f(x,y) = -\frac{p}{\rho\bar{\imath}}\int V(x,y,z)dz + f(x,y), \tag{30}$$

where the function $f(x,y)$ must be chosen so as to ensure that φ is harmonic but is otherwise arbitrary. Since $f(x,y)$ is independent of z, we can determine it from the region $z \gg \lambda$, where

$$-\frac{p}{\rho\bar{\imath}}\int V(x,y,z)dz \to \frac{pz^2}{2} - pbz, \tag{31}$$

from Eq. (20). The condition $\nabla^2\varphi = 0$ can then be satisfied by the choice $f(x,y) = -p(x^2+y^2)/4$, giving

$$\varphi = -\frac{p}{\rho\bar{\imath}}\int V dz - \frac{p(x^2+y^2)}{4} \tag{32}$$

and hence

$$\frac{\partial\varphi}{\partial x} = -\frac{p}{\rho\bar{\imath}}\int\frac{\partial V}{\partial x}dz - \frac{px}{2}. \tag{33}$$

Now on $x = a$, the boundary condition (13) of the electrical problem demands that

$$\frac{\partial^2\varphi}{\partial z\partial x} = 0, \quad \frac{\partial\varphi}{\partial x} = -\frac{pa}{2} \tag{34}$$

for all y,z, from Eqs. (22) and (33). It might be argued that condition (13) only permits us to conclude that on $x = a$

$$\int\frac{\partial V}{\partial x}dz = \frac{\partial}{\partial x}\int V dz = C,$$

where C is any constant. However, the resulting expression must apply for all y,z, including in $z \gg \lambda$, and in this range it is clear that a nonzero value of C would contradict the limiting result (31).

Substituting (34) into Eqs. (3)–(10), we then determine that, on the boundary $x = a$, the lateral displacement u_x is given by

$$u_x = -\frac{p(1-2\nu)a}{4\mu}$$

and the shear tractions $\sigma_{xy} = \sigma_{xz} = 0$ for all y,z. Thus, the only nonzero traction on this surface is the normal traction σ_{xx}. Since the problem is symmetrical with regard to the x and y axes, it is clear that exactly similar results (with appropriate changes of sign and notation) will be obtained on the surfaces $x = -a$ and $y = \pm a$ and hence

$$\sigma_{xx} = g(y,z), \quad u_x = \mp\frac{p(1-2\nu)a}{4\mu}, \quad x = \pm a, \tag{35}$$

$$\sigma_{yy} = g(x,z), \quad u_y = \mp\frac{p(1-2\nu)a}{4\mu}, \quad y = \pm a, \tag{36}$$

where g is an unknown function which tends to the constant

$$g \to -\frac{p(1+2\nu)}{2} \tag{37}$$

in $z \gg \lambda$, from Eqs. (8), (30), and (31).

2. The function g

One way to satisfy the electrical boundary condition (13) would be to impose symmetric periodic boundary conditions on the planes $x = \pm(2n+1)a$ in the infinite layer $-\infty < x < \infty, -\infty < y < \infty, 0 < z < b$, where n is any integer. Since the roughness is statistically random, these symmetry planes and the corresponding y boundaries $y = (2m+1)a$ could be placed in any location or orientation in this infinite layer, from which we conclude that any dependence on y in the function $g(y,z)$ must be statistically random and restricted to wavelengths λ and below.

To explore further properties of the function g, we apply Betti's reciprocal theorem with a simple auxiliary solution comprising unit biaxial tension,

$$\sigma_{xx} = \sigma_{yy} = 1, \quad \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0, \tag{38}$$

for which the corresponding displacements are

$$u_x = \frac{(1-\nu)x}{E}, \quad u_y = \frac{(1-\nu)y}{E}, \quad u_z = -\frac{2\nu z}{E}, \tag{39}$$

excluding an arbitrary rigid-body displacement.

Betti's theorem states that

$$\iint_\Gamma \boldsymbol{t}_1\cdot\boldsymbol{u}_2\, d\Gamma = \iint_\Gamma \boldsymbol{t}_2\cdot\boldsymbol{u}_1\, d\Gamma, \tag{40}$$

where Γ is the boundary of the body and $\boldsymbol{t}_1,\boldsymbol{u}_1$ and $\boldsymbol{t}_2,\boldsymbol{u}_2$ represent the boundary tractions and displacements in two different elastic states. Here we use the index 2 to refer to the auxiliary solution and index 1 to refer to the solution developed from the potential φ.

The only tractions $\boldsymbol{t}_2$ comprise unit tensions on $x = \pm a$ and $y = \pm a$, and the normal component of $\boldsymbol{u}_1$ on these surfaces is given by Eq. (36). Thus the right-hand side of Eq. (40) is

$$-\frac{4p(1-2\nu)(1+\nu)a^2b}{E}$$

since there are four such surfaces, each of area $2ab$, with identical contributions.

Evaluating the left-hand side of Eq. (40), we obtain

$$\frac{8\nu ba^2p}{E} + \frac{8(1-\nu)a^2}{E}\int_0^b\int_{-a}^a g(y,z)dydz,$$

where we note that there are four surfaces $x = \pm a, y = \pm a$, again with identical tractions and displacements. We conclude after some simplification that

$$\int_0^b \int_{-a}^a g(y,z)dydz = -p(1+2\nu)ab. \tag{41}$$

Thus $g(y,z)$ differs from the limiting value (37) only in a region close to $z=0$ and in that region the deviation from the limiting value has a zero average.

3. The corrective solution

The preceding results show that we can off-load the lateral faces to a condition where

$$\int_0^b \int_{-a}^a \sigma_{xx}(a,y,z)dydz = 0 \tag{42}$$

by superposing the auxiliary solution (38) and (39) with the scalar multiplier $p(1+2\nu)/2$. The corrected solution will then satisfy all of the conditions of the original problem except for the existence of a self-equilibrated normal traction in a region close to $z=0$ on the lateral surfaces.

After imposing this correction, the total contact compliance will be

$$w = w_{(a)} + \frac{p(1+2\nu)}{2}\frac{2\nu b}{E} = \frac{2(1-\nu^2)pR_{\mathcal{I}}}{E\rho} + \frac{pb}{E}. \tag{43}$$

Now the second term in this expression defines the plane stress elastic compliance caused by a uniform pressure p if the surface had been smooth, so the additional compliance due to the surface roughness (the interface compliance) is

$$w_{\mathcal{I}} = \frac{2(1-\nu^2)pR_{\mathcal{I}}}{E\rho} \quad \text{or} \quad \frac{1}{R_{\mathcal{I}}} = \frac{2(1-\nu^2)}{E\rho}\frac{dp}{dw_{\mathcal{I}}}, \tag{44}$$

which has the same form as (1) for the case $E_2=\infty, \rho_2=0$. Notice that the differentiation in the second expression in Eq. (44) is legitimate since the contact area $\mathcal{A}$ remains constant during a strictly infinitesimal change in p and hence the incremental contact problem is linear.

V. MORE GENERAL FINITE SYSTEMS

The preceding result is easily extended to the case of two rectangular blocks of different materials and different depths b_1,b_2 since the contact interface $\mathcal{A}$ will be an isopotential surface and will remain planar under deformation provided $\lambda \ll b_1,b_2$. Thus, we can apply the same argument to each body separately, as in Ref. [9], obtaining

$$\frac{1}{R_{\mathcal{I}}} = \frac{2}{E^*(\rho_1+\rho_2)}\frac{dp}{dw_{\mathcal{I}}}, \tag{45}$$

which is the interface resistance equivalent of Eq. (1).

Similar arguments can also be applied to prismatic bodies of general shape contacting on their end faces. The electrical problem then requires that $\partial V/\partial n = 0$ on all lateral boundaries, and an argument parallel to that in Sec. IV A1 can be applied to prove that (i) these boundaries are free of shear tractions based on the equivalence (22) and (ii) the normal tractions on these boundaries can exhibit systematic (nonrandom) dependence only on the coordinate z. Since the plane hydrostatic stress field (38) is invariant with respect to any coordinate transformation within the xy plane, the same corrective procedure as in Sec. IV A1 leads to Eq. (45) as before.

VI. CONCLUSIONS

We conclude that the mathematical analogy between total contact resistance and elastic compliance established in Ref. [9] can be extended to the additional interface resistance per unit area due to surface roughness as in Eq. (45), subject only to the condition that the longest wavelength in the roughness spectrum be small compared with the finite dimensions of the contacting bodies. We emphasize that if this condition is not satisfied, statistical variance associated with the long wavelength terms can be expected to give a wide variance in values of interface resistance for different realizations of the same statistics.

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