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Torsional contact between elastically similar flat-ended cylinders

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ABSTRACT

The problem of frictional contact between two long, elastic, coaxial cylinders pressed together, end-to-end and subsequently subjected to monotonically increasing or oscillating torque is considered and the full state of stress found, together with the partial slip regime and the strength of the assembly.

1. Introduction

Most friction tests are carried out by pressing an indenter onto the surface of a second body by constant normal force and dragging it along. The shear force needed to do so is measured and the ratio of shear to normal force defines the coefficient of friction, f. Unfortunately, this type of test means that the leading edge of the indenter may form a mechanical prow if it is sharp edged and, in any case, as wear proceeds the contacting profile will inevitably be modified. These difficulties are circumvented in a torsion test, because there is no leading edge, and wear will not normally modify the profile of either body. A particularly simple form of the apparatus uses two elastically similar cylinders of the same radius, a, pressed together by a normal force, F, and a torque applied (Gaul and Lenz, 1997). In this paper a detailed analysis of the contact problem is carried out, under both partial slip and spinning conditions. The contact itself is of basic interest because it is so simple; only one length dimension, the cylinder radius, enters the problem if the loading is applied sufficiently remotely, as shown in Fig. 1. The contact is also unusual in that its extent is defined by the dimension of both bodies – any minor misalignment in the plane of the contact will result in only an extremely local disturbance to the stress field derived. In common with a significant number of basic contact mechanics analyses, the contacting surfaces are assumed to be perfectly smooth. Further, we will assume that the loads are such that the macroscopic deformation is purely elastic. Of course there will be some local (asperity scale) plasticity at a real interface. In his 1955 paper (Johnson, 1955), Johnson shows that the elastic Mindlin solution (Mindlin, 1949) for partial slip is a reasonable approximation for real engineering surfaces, However, he remarks that "With an increase of tangential force, elastic distortion alone is not sufficient to secure stress relief, and the asperities at the boundary of the contact surface undergo plastic deformation through quite large strains. This process leads to a sharp increase in energy loss and to marked damage of the surfaces as the metal-to-metal junctions fatigue under sustained vibration". We make the usual assumption that asperity level plasticity is captured by the friction model and in common with a large number of classical contact mechanics solutions we will assume that Amontons/Coulomb friction applies. More sophisticated models might be applied (Burwell and Rabinowicz, 1953; McFarlane and Tabor, 1950), but they would invariably complicate the solution without enhancing insight into the behaviour of this contact geometry. In any case, it will transpire that the contact has the property that it has a separately controllable uniform pressure so that questions of any pressure dependence of friction coefficient do not arise. Indeed, as the contact pressure is uniform it is straightforward to evaluate any pressure-dependent friction effects experimentally without the need for complicated deconvolution processes needed with a Hertzian type contact. The solution will be formed by first looking at the fully adhered (or 'bilateral') solution, where the interface, lying on the $z = 0$ plane plays no rôle. The condition for the onset of slip is found and a perturbation used to determine the resultant stress state subsequently, when a finite slip annulus is present.

2. Bilateral solution

The contact pressure at the interface is obtained by analogy with a single bar, to give

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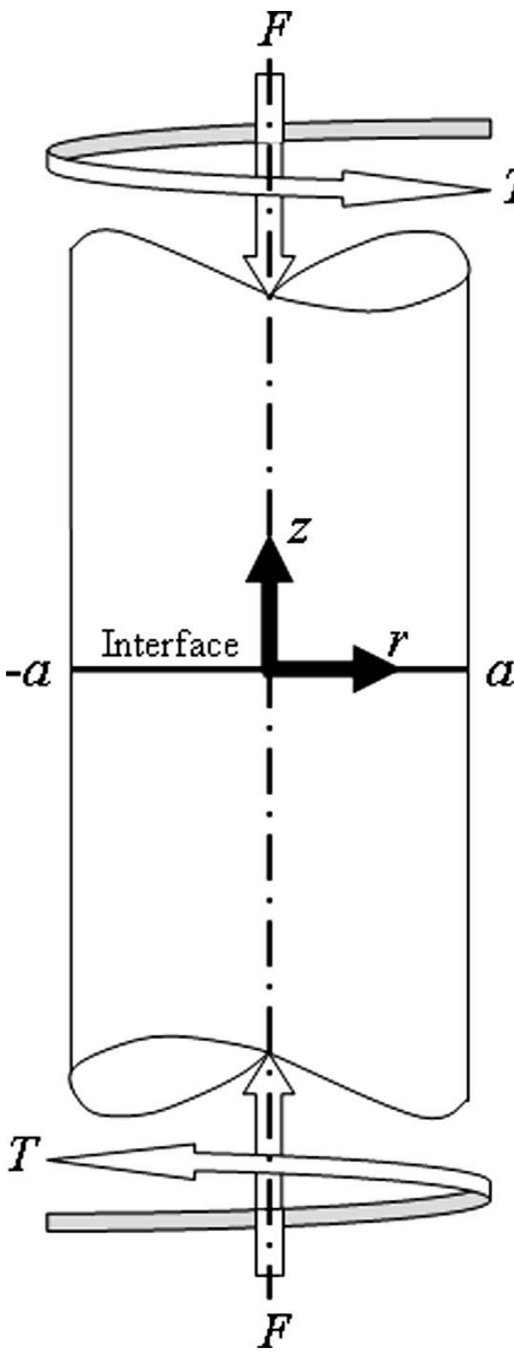

Fig. 1. The torsion problem.

$$\sigma_{zz}(r,0) = -\frac{F}{\pi a^2} = -p_0; \quad 0 < r < a, \tag{1}$$

and, if the coefficient of friction is sufficient to prevent all slip, the two bars together behave as a unitary shaft. The torque, T, induces a (normalized) state of shear stress given by

$$\frac{\sigma_{z\theta}(r,z)a^4}{T} = \frac{2r}{\pi}, \tag{2}$$

$$\sigma_{r\theta}(r,z) = 0, \tag{3}$$

and a θ-direction displacement given by

$$u_\theta = \frac{2Trz}{\pi\mu a^4}. \tag{4}$$

The interface will remain fully adhered provided only that $|\sigma_{z\theta}| < fp_0$ everywhere, hence

$$\frac{|T|}{Fa} < \frac{f}{2}, \tag{5}$$

where f is the coefficient of friction. Note that the maximum torsional shear stress in the bilateral solution (2) occurs at the outer radius and hence slip starts there. Finally, the onset of plastic flow according to von-Mises criterion is when

$$\left(\frac{2T}{\pi a^3 k}\right)^2 + \frac{1}{3}\left(\frac{F}{\pi a^2 k}\right)^2 = 1, \tag{6}$$

where k is the yield stress in pure shear.

3. Formulation: partial slip case

We now assume that the torque has been increased to the point where inequality (5) has been violated, so that there is a central stick disk of radius b within the contact, surrounded by an annulus of slip. The displacement $u_\theta(r,z)$ and associated state of stress are now written down as the sum of the bilateral solution described above (where we now add a superscript, B) and a corrective solution, which will include a superscript C, currently unknown, but which is such that the following conditions may be applied in the contact plane

$$u_\theta(r,0) = u_\theta^B(r,0) + u_\theta^C(r,0) = 0 \quad 0 < r < b \tag{7}$$

$$\sigma_{z\theta}(r,0) = \sigma_{z\theta}^B(r,0) + \sigma_{z\theta}^C(r,0) = \frac{fF}{\pi a^2} \quad b < r < a. \tag{8}$$

$$\sigma_{r\theta} = \sigma_{r\theta}^C(a,z) = 0 \quad \forall z \tag{9}$$

The corrective solution may be formulated using Solution E of Green and Zerna (1968), treated extensively by Barber (1992)

$$2\mu u_\theta^C = -2\frac{\partial\psi}{\partial r}; \quad \sigma_{r\theta}^C = \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\partial^2\psi}{\partial r^2}; \quad \sigma_{z\theta}^C = -\frac{\partial^2\psi}{\partial r\partial z}, \tag{10}$$

where ψ is an axisymmetric harmonic function and μ is the modulus of rigidity. We anticipate that the corrective solution will be local to the interface $z = 0$ and thus we construct the solution as a series of terms of the form

$$\psi = \exp(-\lambda z)g(r), \tag{11}$$

Laplace's equation then reduces to the ordinary differential equation

$$\frac{d^2g}{dr^2} + \frac{1}{r}\frac{dg}{dr} + \lambda^2 g = 0, \tag{12}$$

whose solution, bounded at $r = 0$, is

$$g(r) = AJ_0(\lambda r), \tag{13}$$

in which A is an arbitrary constant and $J_n(.)$ is Bessel's function of order n. We then have

$$\psi = A\exp(-\lambda z)J_0(\lambda r), \tag{14}$$

and substituting into the above expressions, we see that the displacement and stresses induced are given by

$$2\mu u_\theta^C = 2A\lambda\exp(-\lambda z)J_1(\lambda r), \tag{15}$$

$$\sigma_{r\theta}^C = -A\lambda^2\exp(-\lambda z)J_2(\lambda r), \tag{16}$$

$$\sigma_{z\theta}^C = -A\lambda^2\exp(-\lambda z)J_1(\lambda r). \tag{17}$$

The traction-free boundary condition on $r = a$, then requires that

$$\sigma_{r\theta}^C(a,z) = -A\lambda^2\exp(-\lambda z)J_2(\lambda a) = 0, \tag{18}$$

which is equivalent to the requirement that

$$J_2(\lambda a) = 0. \tag{19}$$

The boundary condition (9) is satisfied by this solution if λ satisfies the eigenvalue equation (19) with solution

$$\lambda_n = \frac{\lambda_{2,n}}{a} \quad \text{where } n = 1, 2, 3, \ldots \tag{20}$$

where $\lambda_{2,n}$ are the zeros of the second order Bessel's function, the first few being (Gradshteyn and Ryzhik, 1980)

$\lambda_{2,1} = 5.1356;\quad \lambda_{2,2} = 8.4172;\quad \lambda_{2,3} = 11.6198;$
$\lambda_{2,4} = 14.7960;\quad \lambda_{2,5} = 17.9598.$

Note that some Bessel function relations are given in Appendix A. A sufficiently general solution of the corrective problem can then be constructed as an eigenfunction series

$$\psi = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{\lambda_{2,n}}{a} z\right) J_0\left(\frac{\lambda_{2,n}}{a} r\right) \tag{21}$$

with corresponding stress and displacement components

$$u_\theta^C = \frac{1}{\mu}\sum_{n=1}^{\infty} A_n \frac{\lambda_{2,n}}{a} \exp\left(-\frac{\lambda_{2,n}}{a} z\right) J_1\left(\frac{\lambda_{2,n}}{a} r\right) \tag{22}$$

$$\sigma_{r\theta}^C = -\sum_{n=1}^{\infty} A_n \left(\frac{\lambda_{2,n}}{a}\right)^2 \exp\left(-\frac{\lambda_{2,n}}{a} z\right) J_2\left(\frac{\lambda_{2,n}}{a} r\right) \tag{23}$$

$$\sigma_{z\theta}^C = -\sum_{n=1}^{\infty} A_n \left(\frac{\lambda_{2,n}}{a}\right)^2 \exp\left(-\frac{\lambda_{2,n}}{a} z\right) J_1\left(\frac{\lambda_{2,n}}{a} r\right). \tag{24}$$

In particular, on the end $z = 0$, we have

$$u_\theta^C(r, 0) = \frac{1}{\mu}\sum_{n=1}^{\infty} A_n \frac{\lambda_{2,n}}{a} J_1\left(\frac{\lambda_{2,n}}{a} r\right) \tag{25}$$

$$\sigma_{z\theta}^C(r, 0) = -\sum_{n=1}^{\infty} A_n \left(\frac{\lambda_{2,n}}{a}\right)^2 J_1\left(\frac{\lambda_{2,n}}{a} r\right) \tag{26}$$

and, in order to find the unknown coefficients A_n, the remaining boundary conditions (7) and (8) lead to the dual series equations

$$\sum_{n=1}^{\infty} A_n \frac{\lambda_{2,n}}{a} J_1\left(\frac{\lambda_{2,n}}{a} r\right) = 0;\quad 0 < r < b \tag{27}$$

$$\sum_{n=1}^{\infty} A_n \frac{\lambda_{2,n}^2}{a} J_1\left(\frac{\lambda_{2,n}}{a} r\right) = \frac{2Tr}{\pi a^3} - \frac{fF}{\pi a};\quad b < r < a. \tag{28}$$

Boundary conditions (27) and (28) may now be written in the form

$$\sum_{n=1}^{\infty} \widetilde{A}_n \lambda_{2,n} J_1\left(\frac{\lambda_{2,n}}{a} r\right) = 0,\quad 0 < r < b \tag{29}$$

$$\sum_{n=1}^{\infty} \widetilde{A}_n \lambda_{2,n}^2 J_1\left(\frac{\lambda_{2,n}}{a} r\right) = \frac{r}{a} T^* - 1;\quad b < r < a. \tag{30}$$

where

$$\widetilde{A}_n = \frac{A_n \pi}{fF}, \tag{31}$$

$$T^* = \frac{2T}{fFa}. \tag{32}$$

In the following approach we define a set of weighting functions $J_1\left(\frac{\lambda_{2,m}}{a} r\right)$, $m = 1, \infty$ and proceed to enforce Eqs. (27) and (28) in the weak sense. Hence,

$$\sum_{n=1}^{\infty} \widetilde{A}_n \lambda_{2,n} \left(\int_0^b J_1\left(\frac{\lambda_{2,n}}{a} r\right) J_1\left(\frac{\lambda_{2,m}}{a} r\right) r\, dr + \int_b^a \lambda_{2,n} J_1\left(\frac{\lambda_{2,n}}{a} r\right) J_1 \times \left(\frac{\lambda_{2,m}}{a} r\right) r\, dr \right) = \int_b^a \left(\frac{r}{a} T^* - 1\right) J_1\left(\frac{\lambda_{2,m}}{a} r\right) r\, dr. \tag{33}$$

The eigenvalue problem will exhibit orthogonality so that (Watson, 1958)

$$\int_0^a J_1\left(\frac{\lambda_{2,n}}{a} r\right) J_1\left(\frac{\lambda_{2,m}}{a} r\right) r\, dr = h\left(\frac{\lambda_{2,n}}{a}\right)\delta_{nm}, \tag{34}$$

where

$$h\left(\frac{\lambda_{2,m}}{a}\right) = \frac{a^2}{2}\left\{ [J_0(\lambda_{2,m})]^2 - \frac{2J_0(\lambda_{2,m})J_1(\lambda_{2,m})}{\lambda_{2,m}} + [J_1(\lambda_{2,m})]^2 \right\} \tag{35}$$

the function $h\left(\frac{\lambda_{2,n}}{a}\right)$ is the non-zero value of the integral in the special case $m = n$. It then follows,

$$\int_0^b J_1\left(\frac{\lambda_{2,n}}{a} r\right) J_1\left(\frac{\lambda_{2,m}}{a} r\right) r\, dr = h\left(\frac{\lambda_{2,n}}{a}\right)\delta_{nm} - \int_b^a J_1\left(\frac{\lambda_{2,n}}{a} r\right) J_1\left(\frac{\lambda_{2,m}}{a} r\right) r\, dr, \tag{36}$$

and hence

$$\widetilde{A}_m \lambda_{2,m} h\left(\frac{\lambda_{2,m}}{a}\right) = -\sum_{n=1}^{\infty} \widetilde{A}_n \lambda_{2,n}(\lambda_{2,n} - 1) \times \int_b^a J_1\left(\frac{\lambda_{2,n}}{a} r\right) J_1\left(\frac{\lambda_{2,m}}{a} r\right) r\, dr + \int_b^a \left(\frac{r}{a} T^* - 1\right) J_1\left(\frac{\lambda_{2,m}}{a} r\right) r\, dr. \tag{37}$$

This equation can be written in the following form suitable for solving the simultaneous equations

$$\widetilde{A}_m \lambda_{2,m} h\left(\frac{\lambda_{2,m}}{a}\right) + \sum_{n=1}^{\infty} \widetilde{A}_n . C_{mn} = B_m \tag{38}$$

where expressions B_m and C_{mn} for the integrals terms are given in Appendix B. This provides an infinite set of algebraic equations, similar in form to the set of equations derived by Meleshko and Gomilko (1997) for the rectangle problem (but here only a single set, not a double one, arises).

Although, in reality, the torque is the independent variable and the position of the stick/slip boundary a dependent quantity, in forming a solution we choose a value for b, impose the stick and slip conditions immediately either side this point, solve for the corrective stress distribution and finally evaluate the imposed torque actually present.

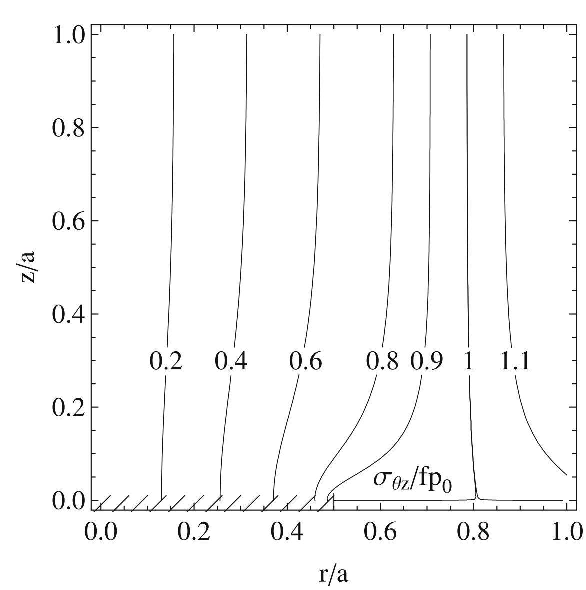

Fig. 2a. Contour plot of the shear stress $\sigma_{\theta z}/fp_0$ for b/a 0.5.

4. Results and discussion

The numerical scheme described above was implemented within commercial code 'MATLAB' but with, of course, the infinite set of simultaneous equations truncated to a finite value, N. It was found that, when N is set to 145 the change in the calculated coefficients of the series is negligible for all values of b/a. An example set of stress and displacement fields is shown in Fig. 2, for the case when $b/a = 1/2$. As may be seen, the $\sigma_{r\theta}$ component of stress which is, of course, absent in the torsion problem, persists for only a short distance from the interface ($|z|/a \sim 1/2$), and that the $\sigma_{\theta z}$ component, associated with torsion, regains its linear variation with r in approximately the same distance. This could have been anticipated from a consideration of the lowest eigenvalue. The most slowly decaying term is proportional to about $\exp\left(-\frac{5z}{a}\right)$ and hence at $z = a/2$ it has fallen to around 0.08 of its value at the interface. Note that the stress components are normalized with respect to the value of the shear traction in the slip annulus, and this is the reason for the $\sigma_{\theta z}$ component exceeding unity in the remote field. The displacement, u_θ has a transient state which persists rather further from the interface, as might be expected.

Fig. 3 shows the size of the stick disk, b/a, as a function of the normalized torque value $T^* = 2T/fFa$. Slip starts when $T^* = 1$, and the 'limit state' spinning condition is when this quantity reaches 4/3. In Fig. 4 we focus on the conditions along the interface, and give values of the shear stresses and circumferential displacement for example values of the applied torque.

We turn, briefly, to a consideration of frictional shakedown. The problem is uncoupled in the sense that θ-direction displacement does not modify the contact pressure, and so, from Barber's recent work on this topic , (Klarbring et al., 2007; Barber et al., 2008) we know that the Melan plasticity shakedown theorem may be applied to a study of the self-generation of residual interfacial shear-

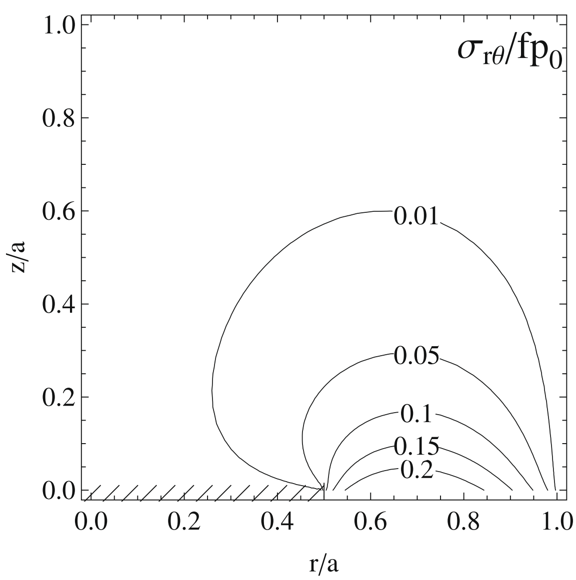

Fig. 2b. Contour plot of the shear stress $\sigma_{r\theta}/fp_0$ for b/a 0.5.

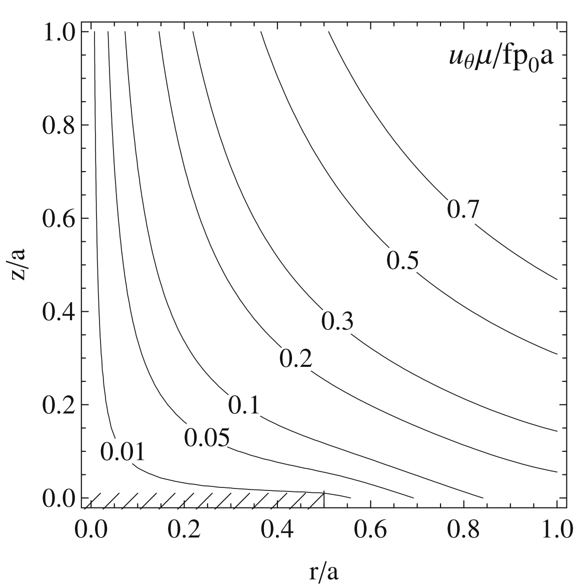

Fig. 2c. Contour plot of the circumferential displacement, u_θ, for b/a 0.5.

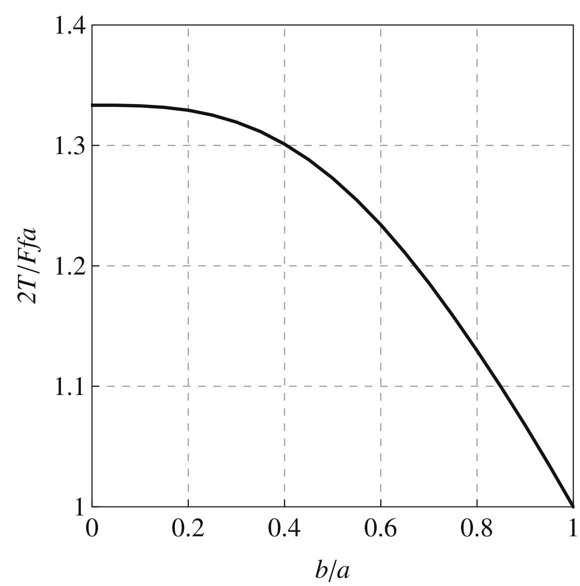

Fig. 3. Torque versus stick disk size, b/a.

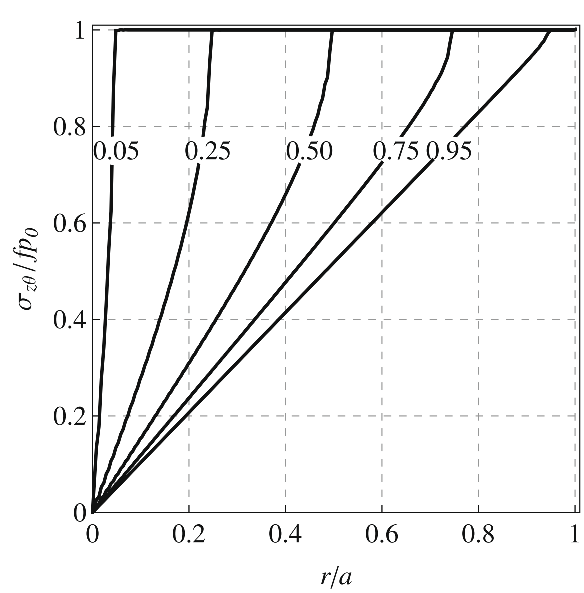

Fig. 4a. Shear traction, $\sigma_{\theta z}$, variation at the contact interface for different values of b/a.

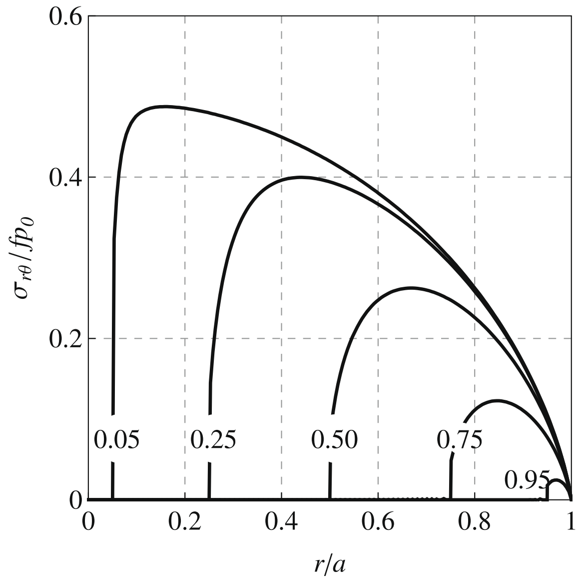

Fig. 4b. Shear stress, $\sigma_{r\theta}$, variation along the contact interface for different values of b/a.

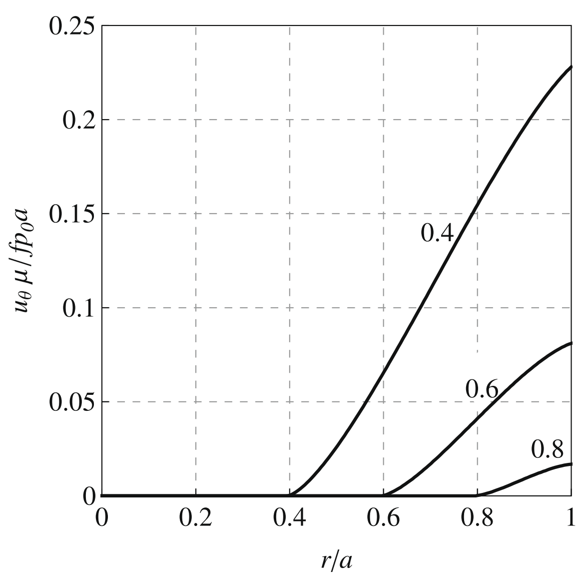

Fig. 4c. Slip displacement, u_θ, variation at the contact surface for different value of b/a.

ing traction. Thus, if the torque oscillates over a range $\Delta T^* = \Lambda$, with a maximum torque $T^*_{\max}$, shakedown to a fully adhered state (within one cycle) is guaranteed provided only that $\Lambda < 2$. In Fig. 5, we display the case of loading, first, to a torque $T^*_{\max} = 1.23$. This corresponds to slip over an annulus leaving a stick disk of radius $b/a = 0.6$ (see Fig. 3). The torque is now released in stages and the net interfacial shearing traction displayed. Note that, upon complete removal of the torque, a distribution of interfacial residual shearing traction left. Upon further decreasing the torque (i.e. increasing it in the opposite sense), the contact remains fully adhered until $T^*_{\min} = -0.77$. Lastly, we note that the presence of slip simply reduces the severity of the state of stress along the interface, and so the elastic limit for a monolithic shaft included in the Introduction applies equally to the split shaft incorporating a frictional interface.

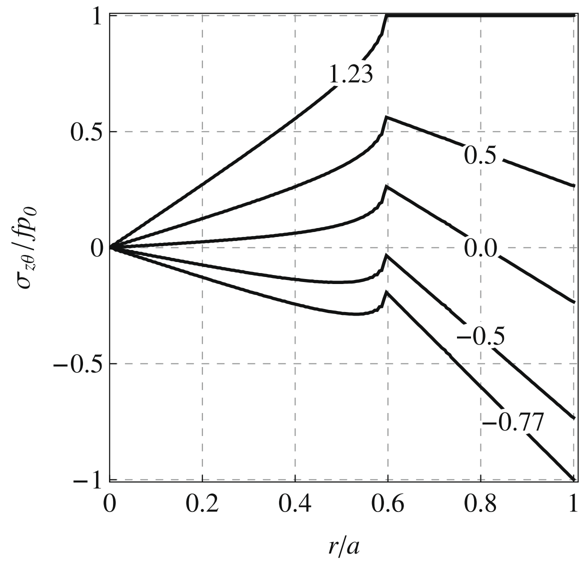

Fig. 5. Cyclic torsion effect. Assembly is loaded to a torque $T^* = 1.23$, giving b/a 0.6 and then gradually relaxed. Note the residual interfacial shearing traction when the assembly is unloaded, and that incipient reverse slip occurs at $T^* = -0.77$.

5. Conclusion

This paper describes an analytical technique for finding the state of stress induced by torsion under partial slip frictional contact conditions, for the case of two cylinders pressed together axially. An accurate representation of the state of stress has been found, and other properties of the problem determined such as the increase in torsional compliance with slip, and its shakedown state under cyclic torsional loading. The solution is obtained under the classical assumption that Coulomb friction applies at a local scale. However, it is recognised that, in real engineering surfaces, local plasticity will inevitably play a role close to the interface and that in some circumstances a different approach may be necessary.

Appendix A. Some Bessel function relations

$$\frac{d}{dx}J_0(x) = -J_1(x) \tag{39}$$

$$2\frac{d}{dx}J_1(x) = J_0(x) - J_2(x). \tag{40}$$

Thus

$$\frac{d^2}{dx^2}J_0(x) = \frac{J_2(x) - J_0(x)}{2}. \tag{41}$$

$$\frac{d}{dr}J_0(\lambda r) = -\lambda J_1(\lambda r) \tag{42}$$

$$\frac{d^2}{dr^2}J_0(\lambda r) = \frac{\lambda^2(J_2(\lambda r) - J_0(\lambda r))}{2}. \tag{43}$$

Also,

$$xJ_0(x) + xJ_2(x) = 2J_1(x), \tag{44}$$

so

$$J_2(x) = \frac{2J_1(x)}{x} - J_0(x) \tag{45}$$

and

$$\frac{d^2}{dx^2}J_0(x)=\frac{J_2(x)}{2}-\frac{J_0(x)}{2}=\frac{J_1(x)}{x}-J_0(x). \tag{46}$$

Thus

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}\right)J_0(x)=\frac{J_1(x)}{x}-J_0(x)-\frac{J_1(x)}{x}=-J_0(x) \tag{47}$$

and

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)J_0(\lambda r)=-\lambda^2 J_0(\lambda r). \tag{48}$$

It follows that

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\lambda^2\right)J_0(\lambda r)=0, \tag{49}$$

as required. for the function to be harmonic. Then

$$\begin{aligned}\frac{1}{r}\frac{d}{dr}J_0(\lambda r)-\frac{d^2}{dr^2}J_0(\lambda r)&=\frac{2}{r}\frac{d}{dr}J_0(\lambda r)+\lambda^2 J_0(\lambda r)\\&=-\frac{2\lambda J_1(\lambda r)}{r}+\lambda^2 J_0(\lambda r).\end{aligned} \tag{50}$$

Appendix B. Integral terms

$$\begin{aligned}B_m&=\int_b^a\left(\frac{r}{a}T^*-1\right)J_1\left(\frac{\lambda_{2,m}}{a}r\right)r\,dr\\&=-\frac{\lambda_{2,m}a^2}{6}\left[{}_1F_2\left(\frac{3}{2};2,\frac{5}{2};-\frac{(\lambda_{2,m})^2}{4}\right)\right.\\&\quad\left.-\left(\frac{b}{a}\right)^3{}_1F_2\left(\frac{3}{2};2,\frac{5}{2};-\frac{b^2(\lambda_{2,m})^2}{4a^2}\right)\right]-\frac{T^*b^2}{\lambda_{2,m}}J_2\left(\frac{b}{a}\lambda_{2,m}\right),\end{aligned} \tag{51}$$

$$\begin{aligned}C_{mn}&=\lambda_{2,n}(\lambda_{2,n}-1)\int_b^a J_1\left(\frac{\lambda_{2,n}}{a}r\right)J_1\left(\frac{\lambda_{2,m}}{a}r\right)r\,dr\\&=\frac{a^2\lambda_{2,n}(\lambda_{2,n}-1)}{(\lambda_{2,m})^2-(\lambda_{2,n})^2}\left[\lambda_{2,n}J_0(\lambda_{2,n})J_1(\lambda_{2,m})-\frac{b}{a}\lambda_{2,n}J_0\left(\frac{b}{a}\lambda_{2,n}\right)J_1\left(\frac{b}{a}\lambda_{2,m}\right)\right.\\&\quad\left.-\lambda_{2,m}J_0(\lambda_{2,m})J_1(\lambda_{2,n})+\frac{b}{a}\lambda_{2,m}J_0\left(\frac{b}{a}\lambda_{2,m}\right)J_1\left(\frac{b}{a}\lambda_{2,n}\right)\right];\\&\quad m\neq n\end{aligned} \tag{52}$$

$$\begin{aligned}&=\frac{a^2}{2}(\lambda_{2,n}-1)\left[\lambda_{2,n}J_0(\lambda_{2,n})^2+\lambda_{2,n}J_1(\lambda_{2,n})^2-2J_0(\lambda_{2,n})J_1(\lambda_{2,n})\right.\\&\quad+2\frac{b}{a}J_0\left(\frac{b}{a}\lambda_{2,n}\right)J_1\left(\frac{b}{a}\lambda_{2,n}\right)\\&\quad\left.-\frac{b^2}{a^2}\lambda_{2,n}J_0\left(\frac{b}{a}\lambda_{2,n}\right)^2-\frac{b^2}{a^2}\lambda_{2,n}J_1\left(\frac{b}{a}\lambda_{2,n}\right)^2\right];\quad m=n.\end{aligned} \tag{53}$$

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