# Energy Conserving Equations of Motion for Gear Systems

**Sejoong Oh**
Senior Engineer
General Motors Corporation, Warren, MI

**Karl Grosh**
Associate Professor

**James R. Barber**
Professor
Department of Mechanical Engineering, The University of Michigan, Ann Arbor, MI

*A system of two meshing gears exhibits a stiffness that varies with the number of teeth in instantaneous contact and the location of the corresponding contact points. A classical Newtonian statement of the equations of motion leads to a solution that contradicts the fundamental principle of mechanics that the change in total energy in the system is equal to the work done by the external forces, unless the deformation of the teeth is taken into account in defining the direction of the instantaneous tooth interaction force. This paradox is avoided by using a Lagrange's equations to derive the equations of motion, thus ensuring conservation of energy. This introduces nonlinear terms that are absent in the classical equations of motion. In particular, the step change in stiffness associated with the introduction of an additional tooth to contact implies a step change in strain energy and hence a corresponding step change in kinetic energy and rotational speed. The effect of these additional terms is examined by dynamic simulation, using a system of two involute spur gears as an example. It is shown that the two systems of equations give similar predictions at high rotational speeds, but they differ considerably at lower speeds. The results have implications for gear design, particularly for low speed gear sets.* [DOI: 10.1115/1.1891815]

## 1 Introduction

The undesirable noise and vibration caused by gears in a large variety of machines has motivated research into uncovering the fundamental noise sources that include variable mesh stiffness, backlash, tooth profile error, misalignment, and surface friction. The negative impact of unwanted noise has driven an industry trend toward more stringent noise specifications. To address these issues, recent research has sought to develop more refined descriptions of vibratory phenomena so that vibration mitigating designs can be developed. In order to obtain more accurate predictions, models are needed that are capable of predicting not only the global behavior, but also higher frequency details of the response. The radiated noise from a gear system, for example, will depend critically on the high frequency response.

A primary source of gear noise and vibration is the varying mesh stiffness. Because the number of gear teeth in contact typically changes during rotation, the mesh stiffness varies with the rotation of gears, introducing a parametric excitation. The deleterious effects of parametric resonance are well known [1,2]. While other vibration sources such as profile errors and contact loss are also extremely important, mesh stiffness variation is unavoidable even in idealized, perfect involute gears and as such serves as the point of departure for this study.

In Den Hartog's book [3], the variation of the number of teeth in contact is modeled as a time-varying stiffness function (a piecewise constant function), a model that is also used in the more recent work of Kahraman et al. [4] and Padmanabhan et al. [5], for example. A constant mesh stiffness model was used in Comparin et al. [6], Padmanabhan et al. [7], and Kahraman et al. [8–10]. In a constant stiffness model, vibrations are excited by either an external force, tooth error, or backlash effects. Comparin et al. [6], Padmanabhan et al. [5,7], and Kahraman et al. [4,8–10] included backlash, where the stiffness depends on the difference between two gears' rotation angles and nonlinearity occurs through loss of contact in this model. Tooth error is an important source of vibration and was studied by Kahraman et al. [4,8,10] and Mark et al. [11], for example.

In all these analyses, a classical Newtonian formulation is used to obtain the equations of motion and the stiffness is assumed to be a known function of time. In practice, of course, the stiffness will be a function of the angular rotation of the gear set and this will only be known as a function of time if the mean rotational speed can be assumed to be (or approximated as) constant. Furthermore, for contacting systems with kinematically varying stiffness, a classical Newtonian statement of the equations of motion leads to a solution that contradicts the fundamental principle of mechanics that the change in total energy in the system is equal to the work done by the external forces, unless the elastic deformation of the contacting bodies is taken into account in defining the direction of the instantaneous contact force [12]. This result is a generalization of Timoshenko's resolution of his paradox regarding the energetics of a vehicle driving over an elastic bridge [13–15].

Two papers have studied perturbations of the mean rotation rate from a constant value. Blankenship et al. [16] investigated the dynamic response of mechanical oscillators for modulation sidebands. They considered stiffness as a function of the displacement, rather than as an explicit function of time. Their model included a linear perturbation of the mean rotation rate by the difference angle and they did not use or develop the coupled dynamical equations for the mean rotation and the deflection. However, their results in part motivate the current study as they found that the effect of an angle modulation on the response is dramatic, even for a small angular perturbation. While Vinayak et al. [17] started their analysis with a consideration of the mesh stiffness dependence on the mean rotation angle (arising from the variations of mean rotation velocity with respect to time), their computations are based on an explicit dependence of the mesh stiffness on time, i.e., the mean rotation angle is a product of constant mean angular velocity with time, uncoupled from the difference in rotation angles. Hence, the coupled evolution equations for mean rotation and relative deflection angles were not derived in either of these studies.

In this paper, we use a Lagrange's equations to derive consistent, energy-conserving equations of motion for a pair of meshing involute spur gears, for which the meshing stiffness is a function of rotational angle. These equations will be shown to differ from the usual Newtonian equations by the inclusion of certain nonlinear terms. Simulation results will then be used to assess the significance of these terms and hence the conditions under which the simpler Newtonian equations provide a reasonable approximation.

## 2 Problem Formulation

**2.1 Equations of Motion.** The gear system is modeled as an inertia-spring system with two degrees of freedom, neglecting damping at outset (Fig. 1). In Fig. 1, $N_1,N_2$ are the numbers of teeth on each gear and $\theta_1,\theta_2$ are the rotation angles of each gear. $T_0$ is a constant applied torque. An input torque on a driving gear is $T_0N_1$ and a balanced output torque on the driven gear is $T_0N_2$,

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although the equations of motion can be derived with arbitrary input and output torques. The generalized coordinates $q_1 = N_1\theta_1$, $q_2 = N_2\theta_2$ are introduced and the following definitions are made:

$$\phi = \frac{q_1 - q_2}{2} = \frac{N_1\theta_1 - N_2\theta_2}{2},$$

$$\theta = \frac{q_1 + q_2}{2} = \frac{N_1\theta_1 + N_2\theta_2}{2},$$

$$q_1 = \theta + \phi, \quad q_2 = \theta - \phi,$$

where $\phi$ is the relative deflection angle between two interacting gears and $\theta$ is the mean rotation angle of two gears. The deformation of the gear teeth is given by $\phi$ while $\theta$ represents the rotational motion for which the gears are designed.

With the definitions above, kinetic energy, $T$, and potential (strain) energy, $V$, of the system can be written

$$T = \frac{1}{2}\frac{I_1}{N_1^2}\dot{q}_1^2 + \frac{1}{2}\frac{I_2}{N_2^2}\dot{q}_2^2 = \frac{I_1}{2}\frac{(\dot{\theta} + \dot{\phi})^2}{N_1^2} + \frac{I_2}{2}\frac{(\dot{\theta} - \dot{\phi})^2}{N_2^2},$$

$$V = \frac{1}{2}k_t(r_1\theta_1 - r_2\theta_2)^2 = 2k_t\left(\frac{r_1}{N_1}\right)^2\left(\frac{N_1\theta_1 - N_2\theta_2}{2}\right)^2$$

$$= \frac{k}{4}(q_1 - q_2)^2 = k\phi^2,$$

$$Q_1 = T_0, \quad Q_2 = -T_0, \quad k = 2k_t\left(\frac{r_1}{N_1}\right)^2, \tag{1}$$

where $k_t$ is translational mesh stiffness between gears (N/mm), $Q_1, Q_2$ are generalized forces, $r_1$ and $r_2$ are the pitch circle radii of each gear, and $I_1, I_2$ are the moments of inertia of each gear. An effective torsional stiffness has been defined for convenience as $k$ (with units of N/mm). Using Lagrange's equation [18] and after some algebra, the equations of motion can be written in terms of the relative deflection angle, $\phi$, and the mean rotation angle, $\theta$, as

$$\ddot{\phi} + M_2 k(\theta)\phi + \frac{M_1}{2}\frac{dk}{d\theta}\phi^2 = M_2 T_0, \tag{2}$$

$$\ddot{\theta} + M_1 k(\theta)\phi + \frac{M_2}{2}\frac{dk}{d\theta}\phi^2 = M_1 T_0, \tag{3}$$

where

$$M_1 = \frac{1}{2}\left(\frac{N_1^2}{I_1} - \frac{N_2^2}{I_2}\right), \quad M_2 = \frac{1}{2}\left(\frac{N_1^2}{I_1} + \frac{N_2^2}{I_2}\right).$$

Equations (2) and (3) define the energy-conserving equations of motion for the gear system of Fig. 1. A more complete gear system, including shafts and bearings, etc., could also be analyzed by the same procedure. However, the principal focus of this study is the gear tooth interaction and additional system parameters are therefore not introduced here in the interest of simplicity.

Notice that nonlinear terms enter into the equations of motion (2) and (3) even though the deflections are assumed to be small. These terms arise through the explicit dependence of mesh stiffness $k$ on the mean rotation angle $\theta$. It is tempting to neglect the terms in Eqs. (2) and (3) involving $\phi^2$ on the grounds that $\phi \ll 1$. However, for spur gears there will be rapid changes of stiffness at the points where the number of teeth in contact changes, leading to instantaneously large values of the multiplier $dk/d\theta$ (Fig. 2).

*2.1.1 Relation to Classical Gear Models*. The classical gear dynamics equations can be viewed as approximations to the energy-conserving equations (2) and (3) based on the neglect of nonlinear effects. We next show which terms must be neglected to obtain the gear equations most commonly used in the literature. One assumption that must be made is that the stiffness $k(\theta)$ is a relatively smooth function of $\theta$. Then we might reasonably drop the $\phi^2$ terms, based on the heuristic argument that they are likely to be negligible. This leads to the equations

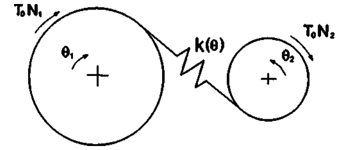


**Fig. 1 Dynamic model for two different gears: $T_0$ is the constant applied torque, $N_1, N_2$ are number of teeth of gears 1 and 2, respectively; $\theta_1$ is the rotation angle of gear 1 and $\theta_2$ is a rotation angle of gear 2; $k(\theta)$ is the meshing stiffness**

$$\ddot{\phi} + M_2 k(\theta)\phi = M_2 T_0, \tag{4}$$

$$\ddot{\theta} + M_1 k(\theta)\phi = M_1 T_0. \tag{5}$$

These are the equations that would be obtained by applying Newton's laws to the system, assuming that the line of action of the tooth interaction force remains at all times on the common tangent to the gear base circles (which, of course, is not true under deformation). This assumption is strictly correct only if the gears are rigid. When tooth deformation is taken into account, the line of action deviates from the ideal line and it is shown in [12] that neglect of this effect in any contact problem with variable stiffness causes the resulting Newtonian equations of motion to violate the energy principle. We shall therefore refer to Eqs. (4) and (5) as the *rigid kinematic Newtonian equations*. They remain nonlinear as long as $k(\theta)$ is considered as a function of $\theta$.

A further approximation that is commonly made is to assume that the stiffness $k$ is a function not of $\theta$, but of time $t$. Strictly, this would only be justified if the mean rotational speed $\dot{\theta} = \Omega_0$ were constant, giving $k(\theta) = k(\Omega_0 t)$, but this clearly contradicts (5), in which the second term is generally nonzero and varies with $t$. However, we note that in the more general case $\theta$ will differ from $\Omega_0 t$ only by a small quantity of order $\phi$ and hence the error involved in replacing $k(\theta)$ by $k(\Omega_0 t)$ will be small, once again as long as $k$ is a smooth function of $\theta$. If we can make this approximation, Eq. (4) reduces to

$$\ddot{\phi} + M_2 k(\Omega_0 t)\phi = M_2 T_0 \tag{6}$$

and is uncoupled from the equation for $\theta$. Equation (6), with all of these approximations, is used in most of the literature on gear vibrations (e.g., [1–9]) and we shall refer to it as the *classical gear equation*.

**2.2 Discontinuities in Stiffness.** The preceding discussion suggests that the classical gear equation (6) is a reasonable approximation of Eqs. (2) and (3) as long as the derivative $dk/d\theta$ is never large in some sense (e.g., $dk/d\theta$ is never orders of magni-

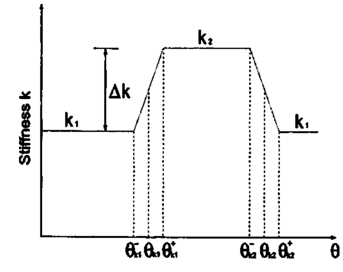


**Fig. 2 Limiting process occurring during the change in the number of teeth in contact**

tude larger than $k$). However, for two meshing spur gears, there is a discontinuity in stiffness at the point where the number of teeth in contact changes, implying a locally infinite value of $dk/d\theta$.

To explore the behavior at such a discontinuity more rigorously, we first consider the piecewise-linear step of Fig. 4, in which the stiffness changes from $k_0 - \Delta k/2$ to $k_0 + \Delta k/2$ over the range $\theta_0 - \epsilon/2 < \theta < \theta_0 + \epsilon/2$—i.e.,

$$k(\theta) = k_0 + \frac{\Delta k(\theta - \theta_0)}{\epsilon}; \quad \theta_0 - \epsilon/2 < \theta < \theta_0 + \epsilon/2. \tag{7}$$

Integrating Eq. (3) with respect to $\theta$ over this interval, we obtain

$$\int_{-\epsilon/2}^{\epsilon/2} \ddot{\theta}\, d\theta + M_1 \int_{-\epsilon/2}^{\epsilon/2} k(\theta)\phi\, d\theta + \frac{M_2 \Delta k}{2\epsilon} \int_{-\epsilon/2}^{\epsilon/2} \phi^2\, d\theta$$
$$= M_1 T_0 \int_{-\epsilon/2}^{\epsilon/2} d\theta. \tag{8}$$

If we now proceed to the limit $\epsilon \to 0$, the second and fourth terms in this equation tend to zero, and the first term can be integrated by parts, giving

$$(\dot{\theta}^{+})^2 - (\dot{\theta}^{-})^2 + M_2 \Delta k \phi^2 = 0, \tag{9}$$

where $\dot{\theta}^{+}, \dot{\theta}^{-}$ are the values of $\theta$ before and after the discontinuity, respectively.

Thus, the nonlinear equations (2) and (3) imply that there will be a discontinuity in rotational speed $\dot{\theta}^2$ across a jump in stiffness, proportional to the instantaneous value of $\phi^2$ and the magnitude of the stiffness change. The gear set will slow down if $\Delta k$ is positive (when the number of teeth in contact increases) and speed up if $\Delta k$ is negative (when the number of teeth in contact decreases).

It is easy to see the energetic cause of this discontinuity in rotational speed. If the stiffness increases for a given value of $\phi$, the elastic strain energy in the gear set increases discontinuously and there is nowhere for this energy to come from other than by a corresponding reduction in the kinetic energy of rotation. Equally, it is clear that both the classical gear equation (6) and the rigid kinematic Newtonian equations (4) and (5), which do not predict such an effect, will lead to spurious discontinuities in the total energy of the system at each discontinuity in stiffness.

Once the change in rotational speed

$$\Delta \dot{\theta} \equiv \dot{\theta}^{+} - \dot{\theta}^{-} \tag{10}$$

has been obtained from Eq. (9), the corresponding change in $\dot{\phi}$ can be found by integrating Eq. (3) across the discontinuity as

$$\Delta \dot{\phi} = \left(\frac{M_1}{M_2}\right) \Delta \dot{\theta}. \tag{11}$$

**2.3 Piecewise Constant Stiffness.** For meshing involute spur gears, the contact stiffness function $k(\theta)$ can reasonably be approximated by a piecewise constant function alternating between two different constant values $k_1, k_2$ depending on the number of teeth in instantaneous contact, as is commonly assumed in gear analysis [19]. Furthermore, it is customary in gear analysis to neglect the contact stiffness (Hertzian-like local deformation) and in the sequel we have done so. For gear sets like those studied in this paper, these assumptions are substantiated by our analysis (not shown in this paper) of the contact problem, following the methods put forth by [20].

During periods of constant stiffness, Eqs. (2) and (3) simplify to the linear equations

$$\ddot{\phi} + M_2 k_i \phi = M_2 T_0, \tag{12}$$

**Table 1 Gear tooth geometry (20° full-depth involute gears)**

| | |
|---|---|
| No. of teeth of gear | 24 (driving gear), 96 (driven gear) |
| Pressure angle | 20° |
| Material of gears | steel (carbon and low alloy) |
| Module | 6.25 (mm) |
| Contact ratio (theoretical) | 1.75 |

$$\ddot{\theta} + M_1 k_i \phi = M_1 T_0, \quad i = 1, 2, \tag{13}$$

which are easily solved analytically. In this case, a piecewise analytic solution algorithm can be devised involving the steps

1. Solve Eqs. (12) and (13) for $i=1$ with given initial conditions for $\theta, \phi, \dot{\theta}, \dot{\phi}$. This will define the values of these quantities as functions of time $t$.
2. Determine the end of the period with $i=1$ by solving the nonlinear equation $\theta(t) = \theta_{12}$, where $\theta_{12}$ is the rotational angle at which the stiffness changes discontinuously from $k_1$ to $k_2$.
3. Use the "jump" equations (9) and (11) with $\Delta k = k_2 - k_1$ to determine the values of $\dot{\theta}, \dot{\phi}$ immediately after the discontinuity. The angles $\theta, \phi$ are assumed continuous across the discontinuity.
4. Solve Eqs. (12) and (13) for $i=2$ with the initial conditions for $\theta, \phi, \dot{\theta}, \dot{\phi}$ from step 3.
5. Determine the end of the period with $i=2$ by solving the nonlinear equation $\theta(t) = \theta_{21}$, where $\theta_{21}$ is the rotational angle at which the stiffness changes discontinuously from $k_2$ to $k_1$.
6. Use the "jump" equations (9) and (11) with $\Delta k = -(k_2 - k_1)$ to determine the values of $\dot{\theta}, \dot{\phi}$ immediately after the discontinuity, thus defining the initial conditions for a repeat of the whole procedure.

## 3 Results and Discussion

The algorithm of Sec. 2.3 was applied to the gear set defined by the parameters given in Table 1. The radii of the pitch circles are $r_1 = 76.2$ mm for the smaller gear and $r_2 = 304.8$ mm for the large gear. The moment of inertia for the smaller gear is $I_1 = 2.89$ kg mm$^2$ while that of the larger gear is $I_2 = 740.2$ kg mm$^2$. All results in this paper pertain to this gear set. Using standard

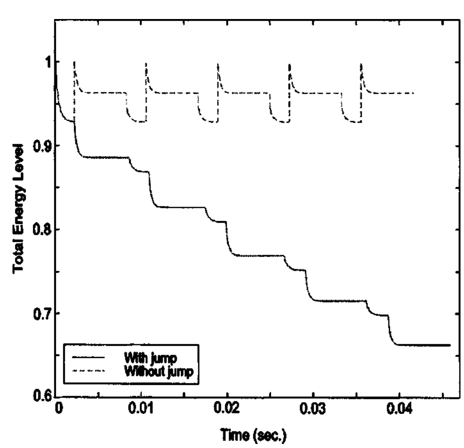


**Fig. 3 Total energy ($U_{total}$) comparison between the energy conserving formulation (with the jump) and the rigid kinematic Newtonian equations (without the jump) using balanced torques**

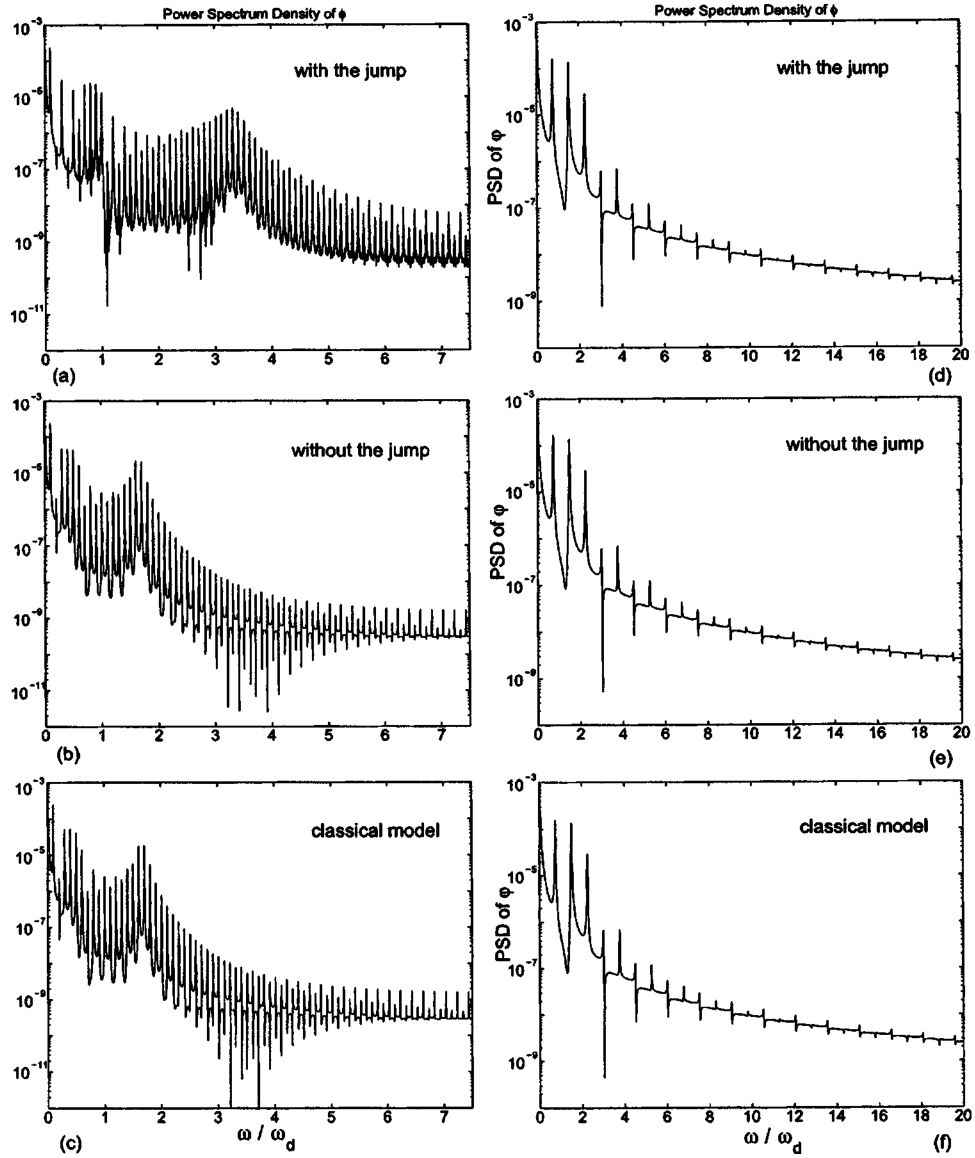


**Fig. 4 Power spectral density of the temporal response: Predictions in (*a*) and (*d*) are made from the equations with the jump, (*b*) and (*e*) are made from the rigid kinematic Newtonian equations (without the jump), and (*c*) and (*f*) are made from the classical model. (*a*)–(*c*) are power spectral densities for an initial angular velocity of the driving gear corresponding to 600 RPM, ($N_1\Omega_1/\omega_d$=0.1). (*d*)–(*f*) are power spectral density for an initial angular velocity of the driving gear corresponding to 4500 RPM ($N_1\Omega/\omega_d$=0.75)**

analysis (e.g., [20]), the piecewise constant torsional stiffnesses were estimated as $k_1=1.43$ kNm with one tooth in contact and $k_2=2.78$ kNm with two teeth in contact. The corresponding angular periods in each state are $\theta_{12}=1.73$ rad and $\theta_{21}=4.56$ rad, respectively.

A small amount of viscous damping was introduced into the piecewise linear computations. For simplicity, we have taken a form of the damping force that is proportional to $\dot{\phi}$. The damping force term that is then included on the left-hand side of Eq. (2) is $c(r_1/N_1)^2M_2\dot{\phi}$. A dimensionless viscous damping factor in the range $0.05<\zeta<0.1$ is typical for interacting gears [21–23] and is used in this study, where we define

$$\zeta = \frac{c}{2}\sqrt{\frac{1}{k_0M_2}}, \quad k_0 = \frac{k_1+k_2}{2} \tag{14}$$

and $c$ is the damping coefficient.

**3.1 Energy Considerations.** We define the total energy $U$ as

$$U = T + V - W, \tag{15}$$

where $T$ is the kinetic energy of the rotating shafts, $V$ is the elastic strain energy in the spring $k(\theta)$, and $W$ is the work done by the external torques. In the absence of damping, $U$ should remain constant, while with damping it can only decrease with time $t$.

Figure 3 shows the total energy $U$ as a function of $t$ for the gear set of Fig. 3 for $\zeta=0.1$ and an initial condition in which the driving 24-tooth gear rotates at 300 rpm. The solid curve represents the predictions of Eqs. (2) and (3) implemented through the algorithm of Sec. 2.3, while the dotted curve was obtained using Eqs. (4) and (5), for which no jump in $\theta$ is obtained at the discontinuities in stiffness. Not surprisingly, the results from the energetically correct equations exhibit a monotonically decreasing total energy due to viscous dissipation, but when the $\dot{\phi}^2$ terms in these equations are neglected, physically unrealistic upward jumps in total energy occur when the number of teeth in contact increases, leading to a paradoxical situation where viscous damping is present, but there is no overall reduction in total energy from cycle to cycle.

**3.2 Dynamic Response.** The most critical question is “To what extent do the approximations inherent in the classical gear equation affect the noise and vibration performance of the gear set?” Figure 4 shows the power spectral density (PSD) of the temporal response of $\phi$ at two different rotational speeds, with a damping factor of $\zeta=0.05$. In each case, results are presented based on (i) the exact energy-conserving equations (2) and (3), (ii) the rigid kinematic Newtonian equations (4) and (5), and (iii) the classical gear equation (6). The frequency $\omega$ is normalized by the damped natural frequency of the system

$$\omega_d = \sqrt{M_2 k_0 (1-\zeta^2)} \tag{16}$$

at the mean tooth stiffness $k_0$.

When the driving gear rotates at 4500 rpm, all three sets of equations define essentially the same response, hence justifying the use of the simpler classical equation (6) for vibration analysis. However, with the driving gear rotating at 600 rpm, the exact results differ markedly from the predictions of both Eqs. (4) and (5) and Eq. (6). In particular, the exact equations predict a peak response at $\omega \approx 3.2\omega_d$ which is completely absent from the approximate results. The amplitude of this high frequency noise is nearly 400 times larger than that predicted by the classical gear equation.

Even at 600 rpm, Figs. 4(*b*) and 4(*c*) are almost identical, showing that the errors in the classical theory are largely due to the neglect $\phi^2$ terms in (2) and (3) and, in particular, the resulting jumps predicted by (9) and (11).

## 4 Conclusions

Using a Lagrange’s equation to formulate the gear interaction problem, we have demonstrated that nonlinear terms must be included in the equations of motion if the resulting predictions are to satisfy the fundamental mechanics principle that the work done by the external forces be equal to the change in total energy in the system. We show that the classical gear equations predict an artifical net increase in energy during one cycle of rotation. These nonlinear terms become significant at points where the tooth contact stiffness changes rapidly, in particular when the number of teeth in contact changes between two meshing spur gears.

Dynamic simulation results show that the importance of these effects depends on the rotational speed of the system. At high rotational speeds, the frequency response is indistinguishable from that predicted by the classical equations, but at slower speeds significant differences are observed. In particular, the exact energy-conserving equations predict a peak in high frequency gear vibration that is not present in the frequency response of the approximate analysis.


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