

# Transient Contact of Two Sliding Half-Planes With Wear

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*We study the transient contact of two sliding bodies with a simple geometry. The model employs the Archard law of wear in which the rate of material removal is proportional to pressure and speed of sliding. The problem is formulated in terms of two governing equations with unknown pressure and heat flux at the interface. The equations are solved numerically, using appropriately chosen Green's functions. We start with a single area of contact. As a result of frictional heating and thermal expansion, the contact area shrinks, which leads to further localization of pressure and temperature. The role of wear is twofold. By removing protruding portions of the two bodies, wear tends to smoothen out pressure and temperature. On the other hand, it causes the contact area to grow sufficiently large to become unstable and bifurcate. Areas carrying load are eventually removed by wear, and the contact moves elsewhere. The system develops a cyclic behavior in which contact and non-contact areas interchange.*

## Introduction

Temperature and pressure development at the contacting surfaces of two bodies can be unstable in the sliding process involving frictional heating. This phenomenon, known as thermoelastic instability, has been observed in many practical sliding systems [1-4]. If wear is absent and one body is a rigid insulator, the process tends to a known steady state [5, 6]. Instability of transient contact for nominally conforming surfaces was studied in [7, 8, 9]. It was shown that there exists some critical value of sliding speed,  $V_{cr}$ , for which the pressure, if started with a sinusoidal disturbance, remains unchanged. For  $V < V_{cr}$  the disturbance decays and otherwise it grows without limit. For  $V > V_{cr}$ , the superimposed pressure eventually becomes insufficient to keep the two bodies in contact throughout the nominal contact area and patch-like contact develops.

The transient development of patch-like contact in the absence of wear was treated in [10], using the Green's function representation of [11]. The presence of wear complicates the process. It has been shown for related problems [12, 13] that in some cases wear damps the growth of disturbances, while in other cases wear in itself gives rise to oscillatory behavior.

In this paper, we use the Green's function representations of [11] to study the contact problem of two sliding half-planes with wear. We modify the technique of [10], which allows us to relax some of the earlier restrictions. Numerical results show that the system develops a cyclic behavior in which contact and separation areas interchange periodically.

**The Model.** The model considered is shown in Fig. 1. Two bodies are pressed together by force  $P$ . One of the bodies slides on the surface of the other and heat is generated due to friction. The surface of body 1 is slightly rounded to give an

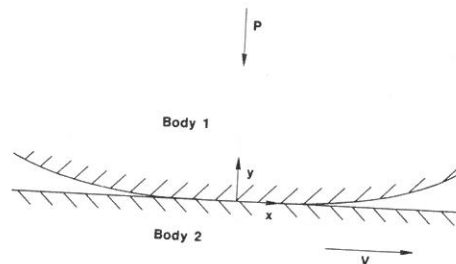


Fig. 1 Model for surface contact

initial Hertzian pressure distribution. This also ensures that the contact area will remain substantially stationary with respect to body 1 and move at the sliding speed  $V_0$  over body 2. Both bodies are elastic and thermal conductors. Shearing tractions on the surface are assumed to be proportional to the normal pressure, with coefficient of friction  $\mu$ . In solving the contact problem, we take into account the coupling effect between normal and shear tractions. Wear is assumed to follow Archard's law [14] according to which the rate of material removal,  $w$ , is proportional to pressure and speed of sliding. In the heat conduction problem, we assume that temperatures of the two bodies are equal at the interface in the contact region, and there is no heat flux across the surfaces in the separation region.

**Governing Equations and Boundary Conditions.** One way to treat the problem is to write down the governing integral equations in terms of appropriately chosen Green's functions. We use the fundamental solution corresponding to the release at time  $t = 0$  of a unit quantity of heat per unit length in  $z$  direction at  $x = 0, y = 0$  on the surface of a half-plane. For body 1 the temperature is given by [15]<sup>1</sup>

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<sup>1</sup>Equation (1) differs from the result in [15] by a factor of 2, since the latter is the solution for a point source in an infinite body, whereas in the present problem all the heat passes into the half-plane  $y > 0$ .

$$T = \frac{\exp(-R^2)}{2\pi k_1 \rho_1 c_1 t} \quad (1)$$

where

$$R^2 = \frac{x^2 + y^2}{4k_1 t} \quad (2)$$

The corresponding normal displacements on the traction free surface are [11]

$$u_y = -\frac{\alpha_1(1+\nu_1)}{\pi \rho_1 c_1 \sqrt{k_1 t}} \Phi_1(R) \quad (3)$$

In equations (1) and (3)  $\alpha_1$ ,  $k_1$ ,  $\rho_1$ ,  $c_1$ , and  $\nu_1$  are, respectively, the coefficient of thermal expansion, thermal diffusivity, density, specific heat, and Poisson's ratio for the material of the first body. The function  $\Phi_1(R)$  is related to the complex error function discussed by Miller and Gordon [16]. Numerically efficient series for small and large arguments are given in [11].

If we fix the coordinate system for the first body, fundamental solutions for temperature and normal displacements of the second body will be

$$T = \frac{\exp(-B^2)}{2\pi k_2 \rho_2 c_2 t} \quad (4)$$

$$u_y = -\frac{\alpha_2(1+\nu_2)}{\pi \rho_2 c_2 \sqrt{k_2 t}} \Phi_1(B) \quad (5)$$

where

$$B^2 = \frac{(x-Vt)^2 + y^2}{4k_2 t} \quad (6)$$

Using these results, the system of governing equations for the unknown pressure distribution and temperature on the surface can be written

$$\begin{aligned} u(x,t) = & \frac{2}{\pi} \left( \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \right) \int_{A(t)} p(x_1,t) \log|x-x_1| dx_1 \\ & + \frac{\mu}{2} \text{sign}(V) \left[ \frac{(1-2\nu_1)(1+\nu_1)}{E_1} \right. \\ & \left. - \frac{(1-2\nu_2)(1+\nu_2)}{E_2} \right] \int_{A(t)} p(x_1,t) \text{sign}(x_1-x) dx_1 \\ & - \frac{\alpha_1(1+\nu_1)}{\pi \rho_1 c_1} \int_0^t \int_{A(t_1)} \frac{q_1(x_1,t_1)}{\sqrt{k_1(t-t_1)}} \Phi_1(R_1) dx_1 dt_1 \\ & - \frac{\alpha_2(1+\nu_2)}{\pi \rho_2 c_2} \int_0^t \int_{A(t_1)} \frac{q_2(x_1,t_1)}{\sqrt{k_2(t-t_1)}} \Phi_1(B_1) dx_1 dt_1 \\ = & c + \frac{x^2}{2r} - V \int_0^t w p(x,t) dt_1; \quad x \in A(t) \end{aligned} \quad (7)$$

$$\begin{aligned} T_1(x) = & \frac{1}{2\pi k_1 \rho_1 c_1} \int_0^t \int_{A(t_1)} \frac{q_1(x_1,t_1) \exp(-R_1^2) dx_1 dt_1}{(t-t_1)} \\ = & T_2(x) = \frac{1}{2\pi k_2 \rho_2 c_2} \int_0^t \int_{A(t_1)} \\ & \frac{q_2(x_1,t_1) \exp(-B_1^2) dx_1 dt_1}{(t-t_1)}; \quad x \in A(t) \end{aligned}$$

$$q_1 + q_2 = \mu p V$$

where

$$R_1 = \frac{x-x_1}{2\sqrt{k_1(t-t_1)}} \quad B_1 = \frac{x+V(t-t_1)-x_1}{2\sqrt{k_2(t-t_1)}} \quad (10)$$

In equations (7)–(9),  $u(x, t)$  is the relative surface displacement of the two bodies,  $p$  is the contact pressure,  $q_1$  and  $q_2$  are the heat fluxes directed into bodies 1, 2, respectively, and  $w$  is coefficient in Archard's wear law. Equation (7) states that the two bodies are in contact in the region  $A(t)$ . In particular, the first two integrals in (7) give the contribution of elastic displacements caused by the pressure and frictional (tangential) stresses, and the next two integrals take into account thermal displacements caused by frictionally generated heat flows  $q_1$  and  $q_2$  across the surfaces. The first right hand side term is a mismatch of the initial surface profiles and the last term shows the contribution of wear to the surface geometry. The constant,  $c$ , represents a relative rigid body displacement of the two bodies. Expression (8) states that temperature is continuous at the interface. Finally, (9) states that the total heat flux into both bodies is equal to the frictionally generated heat.

The integrals in (7, 8) should be performed over the domain of contact  $A(t)$ ; this is also the range of definition of the integral equations. The contact area may be simply or multiply connected and is unknown in advance. It is defined by conditions of no-tensile-pressure

$$p(x,t) = -\sigma_{yy}(x,0,t) \geq 0, \quad x \in A(t), \quad (11)$$

no-interpenetration

$$u_y(x,t) \geq c - \frac{x^2}{2r} - V \int_0^t w p(x,t_1) dt_1 \quad x \in A(t), \quad (12)$$

and equilibrium

$$\int_{A(t)} p(x,t) dx = P(t) \quad (13)$$

In equation (12),  $r$  is the initial radius of curvature of body 1 and  $c$  is a rigid body displacement. As we mentioned earlier, one of the surfaces is assumed to be rounded, but we could take any other surface which suits our purpose. In equation (13),  $P(t)$  is the prescribed force pressing the two bodies together, and  $A(t)$  is the part of the surface out of contact.

**Dimensionless Formulation.** To reduce the number of parameters, we write

$$x = A_0 x^* \quad (14)$$

$$T = T^* \frac{\mu P V}{k_1 \rho_1 c_1} \quad (15)$$

$$p = p^* \frac{P}{A_0} \quad (16)$$

$$q = q^* \frac{\mu P V}{A_0} \quad (17)$$

$$t = t^* \frac{A_0^2}{k_1} \quad (18)$$

$$A_0 = \frac{3\pi(1-\nu_1)k_1\rho_1c_1}{4\alpha\mu VE} \quad (19)$$

The effective Young's modulus is defined by

$$E = \frac{E_1 E_2 (1-\nu_1^2)}{E_2(1-\nu_1^2) + E_1(1-\nu_2^2)} \quad (20)$$

- (8) and dimensionless quantities are denoted by asterisks. If one body is an insulator and wear is absent,  $A_0$  in (19) becomes the half-width of the area of contact in the steady state [5]. This is the motivation for using  $A_0$  as a reference length dimension.

Substituting these into equations (7, 8, 13) and dropping asterisks, yields

$$\begin{aligned}
 & \int_{A(t)} (q_1 + q_2) \log |x - x_1| dx_1 \\
 & + A_5 \int_{A(t)} (q_1 + q_2) \text{sign}(x_1 - x) dx_1 \\
 & - \frac{3\pi}{8} \left\{ \int_0^t \int_{A(t)} \frac{q_1(x_1, t_1)}{\sqrt{t-t_1}} \Phi_1 \left( \frac{x-x_1}{2\sqrt{t-t_1}} \right) dx_1 dt_1 \right. \\
 & + \frac{A_2}{A_3} \int_0^t \int_{A(t)} \frac{q_2(x_1, t_1)}{\sqrt{A_1(t-t_1)}} \times \\
 & \left. \Phi_1 \left[ \frac{x-x_1 + A_6(t-t_1)}{2\sqrt{A_1(t-t_1)}} \right] dx_1 dt_1 = \right. \quad (21) \\
 & = c + x^2/A_4^2 - A_7 \int_{A(t)} (q_1 + q_2) dt; \\
 & A_1 \int_0^t \int_{A(t)} q_1(x_1, t_1) \frac{\exp \left[ -\frac{(x-x_1)^2}{4(t-t_1)} \right] dx_1 dt_1 =
 \end{aligned}$$

$$= A_2 \int_0^t \int_{A(t)} \frac{q_2(x_1, t_1) \exp \left[ -\frac{(x-x_1 + A_6(t-t_1))^2}{4A_1(t-t_1)} \right] dx_1 dt_1}{(t-t_1)} \quad (22)$$

$$\int_{A(t)} (q_1 + q_2) dx = 1$$

where

$$A_1 = k_2/k_1 \quad (24)$$

$$A_2 = \frac{\rho_1 c_1}{\rho_2 c_2} \quad (25)$$

$$A_3 = \frac{\alpha_1(1+\nu_1)}{\alpha_2(1+\nu_2)} \quad (26)$$

$$A_4 = A(0)/A_0 \quad (27)$$

$$A_5 = 0.5 \pi \mu \beta \text{sign}(V) \quad (28)$$

$$A_6 = A_0 \frac{V}{k_2} \quad (29)$$

$$A_7 = \frac{\pi w A_0 E V}{2k_1(1-\nu_1^2)} \quad (30)$$

and the initial half-width of the contact area is

$$A(0)^2 = \frac{4(1-\nu_1^2)Pr}{\pi E} \quad (31)$$

Parameter  $A_5$  is related to Dundur's constant [18]

$$\beta = \frac{E_2(1-2\nu_1)(1+\nu_1) - E_1(1-2\nu_2)(1+\nu_2)}{E_2(1-\nu_1^2) + E_1(1-\nu_2^2)} \quad (32)$$

and describes the coupling between tangential and normal tractions. We note that there is no coupling for two similar or incompressible materials or if one body is incompressible and

the other is relatively rigid (e.g., rubber on steel). The rigid body displacement  $c$  is not a parameter, but has to be found as part of the solution.

In general, the solution depends on the seven dimensionless parameters  $A_1 - A_7$ , but in many important cases the number of parameters is substantially reduced. For example, if one body is a rigid nonconductor and the coupling between normal and tangential tractions can be neglected, in the absence of wear the problem depends on one parameter [10]. For two similar materials,  $A_1 = A_2 = A_3 = 1$ ,  $A_5 = 0$  and the solution depends on three parameters.

**Numerical Implementation.** The problem is discretized in space by dividing the contact area into many strips and representing the pressure, heat flux and temperature by polygons. These polygons are made by combining triangular shape functions which are used as numerical Green's functions. These in turn are derived by integration of (1) and (3) for linearly distributed heat flow. Results for the displacements can be found in [10], and for temperature they can be derived from those given in [15] for a continuous point source of unit strength; i.e.,

$$T = -\frac{Ei \left( -\frac{x^2}{4kt} \right)}{2\pi k \rho c} \equiv -\frac{Ei(-X^2)}{2\pi k \rho c} \quad (33)$$

where  $-Ei(-X) = \int_X^\infty \frac{e^{-u}}{u} du$  is the exponential integral.

For a linearly distributed source  $q(x) = q_1 x$  results are obtained by integration, giving

$$T = -\frac{q_1}{2\pi k \rho c} \int_0^x x_1 Ei \left[ -\frac{(x-x_1)^2}{4kt} \right] dx_1 = -\frac{2q_1 t}{\pi \rho c} \phi_2(X) \quad (34)$$

where

$$\phi_2(X) = \frac{1 - \exp(-X^2)}{2} + \frac{X^2}{2} Ei(-X^2) - \sqrt{\pi} X \text{erf}(X) \quad (35)$$

The problem is discretized in time by replacing the actual distribution of heat flux and pressure by a piece-wise constant representation. The contact area and pressure are updated at the end of each time step on the basis of accumulated thermoelastic displacements. The unknowns are the heat fluxes  $q_1$  and  $q_2$ , the temperatures at each node in the contact area, and the rigid body displacement  $c$ . Algebraic equations corresponding to (23, 24) are written for all nodes. One more equation comes from (25). The contact area is unknown in advance and is found by iteration. If it is not guessed correctly, negative pressure develops at some points and overlapping at some others. We then release points of the first type and introduce contact for points of the second type, and repeat the procedure. The convergence of this algorithm is demonstrated in [17]. It proves to be numerically efficient. Experience shows that no more than ten iterations were required even for multiply-connected areas of contact.

We basically follow a numerical procedure similar to that described in [10]. However, generalization of the problem leads to a substantial increase in the volume of computations. When both bodies are conductors, equations (23, 24) are coupled and symmetry about the mid-point is not preserved. In addition, wear introduces new features into the process which makes it desirable to compute many more time steps. Because of this, we have to pay serious attention to numerical efficiency of implementation. For example, the computations became numerically unstable if a triangular heat input is far away in space or time from the node of interest [17]. To overcome this problem and reduce computational time, the cor-

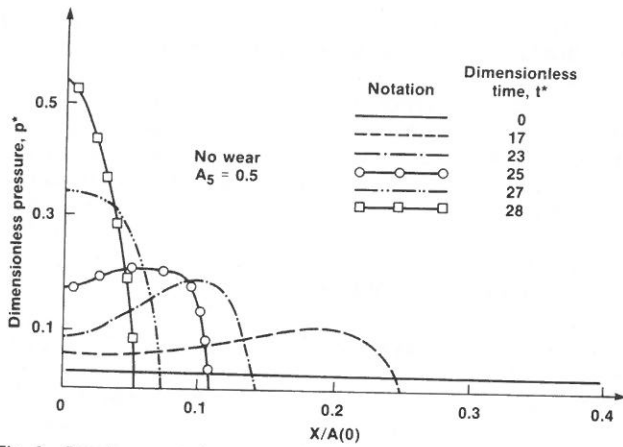


Fig. 2 Development of pressure distribution for  $A_4 = 25$ , no coupling between normal and tangential tractions

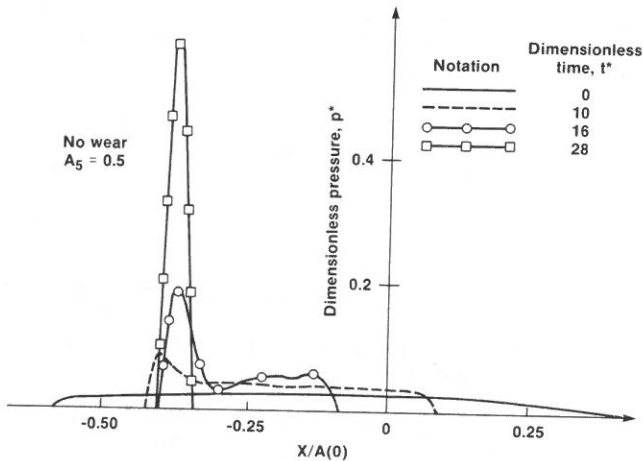


Fig. 3 Development of pressure distribution for  $A_4 = 25$ . Coupling between normal and tangential tractions is included.

responding Green's functions are replaced by those for a line or point source. We also sacrificed a variable time step. This has the advantage that when we go to the next time step, we have to compute Green's functions only for the most remote time step, all the others being already available.

## Results

The algorithm described in the previous section was implemented and used to explore the behavior of the system with various values of the parameters.

We first consider cases where body 2 is an insulator, since the behavior of the system is then simpler, depending only on the parameters  $A_4$ ,  $A_5$ ,  $A_7$  and comparison can be made with the previous results. In the previous analysis [10, 17], coupling between normal and tangential tractions was neglected ( $A_5 = 0$ ) and there was no wear ( $A_7 = 0$ ). The results therefore depended only on the single parameter  $A_4$ . For  $A_4$  less than some critical value ( $\approx 28$ ), the contact area was found to shrink smoothly until the steady-state size ( $A_0$ ) was reached. For larger values of  $A_4$ , wavy perturbations in the contact pressure distribution grew sufficiently to cause bifurcation of the contact area, which eventually gave way to a single-connected contact area in the steady-state. Development of the pressure distribution is reproduced here in Fig. 2, for reference.

Figure 3 shows the effect of introducing coupling between the normal and tangential tractions ( $A_5 = .5$ ), still in the absence of wear. We see that the pressure distribution still exhibits the same qualitative behavior, tending continuously to a

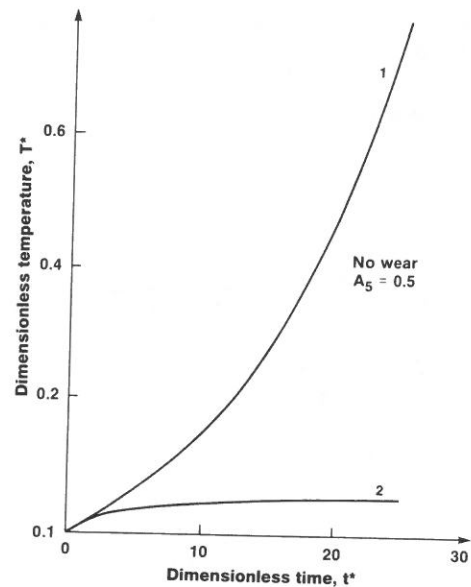


Fig. 4 Development of the maximum temperature at the surface for  $A_4 = 25$ : 1 — one plane is an insulator, 2 — two similar materials.

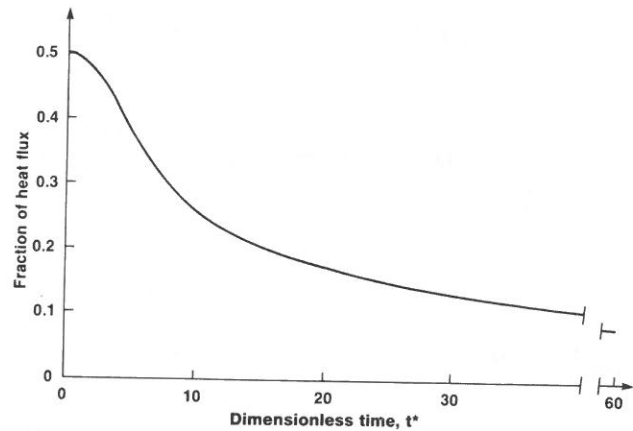


Fig. 5 Fraction of the heat flux across the surface of the stationary body

steady-state with a reduced contact area, but the coupling destroys the symmetry of the system and causes the pressure peak to be displaced to the upstream side of the origin.

The variation of maximum temperature with time is shown in Fig. 4, curve 1. This does not tend to a steady-state, since the half-plane heated in a finite region of its boundary does not have a bounded steady-state solution. However, if we relax the condition that body 2 is an insulator, we find that the temperature also approaches a steady-state (curve 2 in Fig 4). This can be explained by noting that the moving body is continually presenting new cool material to the contact region. As the temperature in the contact region increases, the proportion of frictional heat conducted into body 2 increases. Eventually a condition will be reached where all the heat is conducted away through the moving body, 2, and the distribution of flux into body 1 becomes self-equilibrating. This is illustrated in Fig. 5, for the case of similar materials. In this case, the frictional heat is initially equally divided between the bodies, but the proportion flowing into body 1 is seen to fall steady with time.

When wear occurs, there can be no steady-state, since the thermal bulge must eventually be worn off, transferring the load to a different region. In fact there is ample experimental evidence that this leads to a more or less random process, in

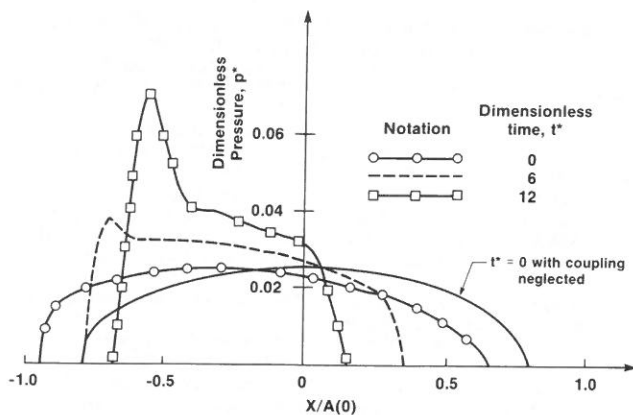


Fig. 6 Development of pressure distribution with time for  $A_4 = 25$  in the presence of wear, earlier stage

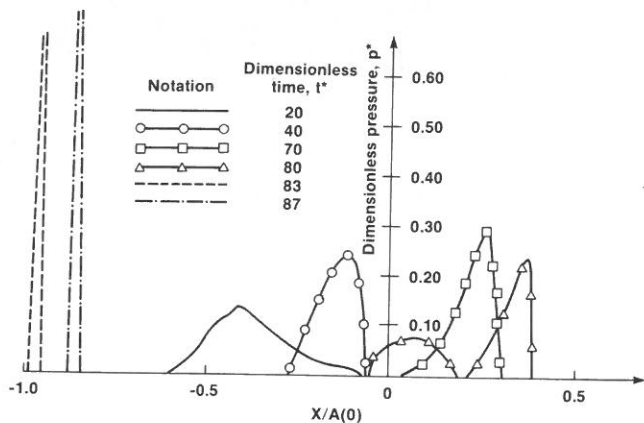


Fig. 7 Development of pressure distribution with time for  $A_4 = 25$  in the presence of wear, later stage

which the contact region moves periodically from place to place over the nominal contact area, giving transient high temperature excursions [2].

This effect is illustrated by two examples in Figs. 6–9, for which again we take body 2 to be an insulator in the interests of simplicity. Figures 6 and 7 show the development of the contact pressure distribution for  $A_4 = 25$ ,  $A_5 = .5$  and  $A_7 = 2$ . These values are appropriate for ceramic/metal interfaces at high sliding speeds [19, 20]. In the early stages, the pressure profile follows the same pattern as in the zero wear case—Fig. 3. The contact area shrinks and the maximum pressure is displaced upstream as a consequence of normal/tangential coupling. However, bifurcation of the contact area then occurs, after which an additional contact area appears at the extreme upstream end of the original contact area. This new region rapidly takes over the entire load, leading to a very high contact pressure, after which it starts to move back across the previously worn area.

Changing the dimensionless initial contact width,  $A_4$ , to 20 produces the results of Figs. 8 and 9. Again, the early behavior is similar to that of Fig. 3, but this time a more chaotic pattern is developed, with alternation between periods of two contact areas and a single area. Maximum pressures tend to increase with the passage of time, but are lower than those achieved in the previous example.

We emphasize that the initial contact width for both the above examples would have given no bifurcation in the absence of wear [10, 17].

## Conclusions

The introduction of wear into the analysis of the transient thermoelastic contact of two sliding half-planes, leads to some

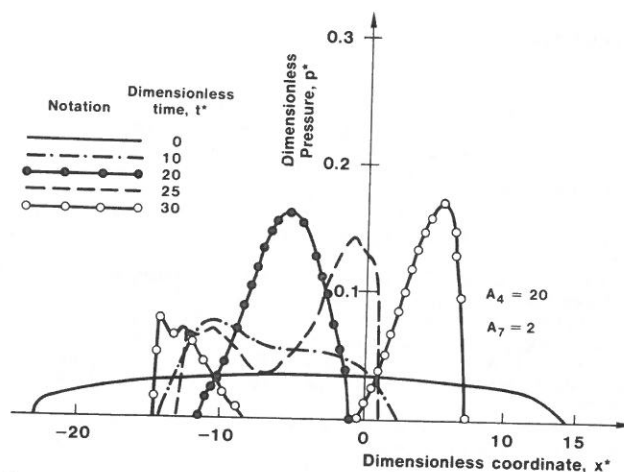


Fig. 8 Development of pressure distribution with time for  $A_4 = 20$  in the presence of wear, earlier stage

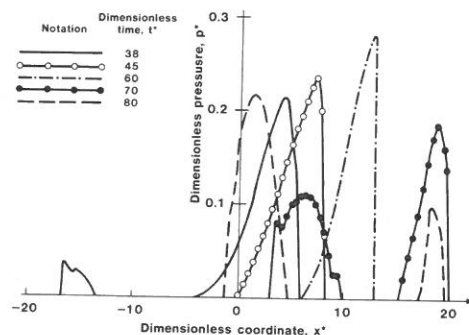


Fig. 9 Development of pressure distribution with time for  $A_4 = 20$  in the presence of wear, later stage

unexpected results. It is natural to see wear as removing material from the highest parts of the surface and hence smoothing out the pressure distribution—indeed, Dow and Burton [12] have shown that wear has a stabilizing effect on thermoelastic contact in that it increases the critical speed required for thermoelastic instability to occur.

However, the results of this analysis show that moderate amounts of wear have a destabilizing effect on the development of a steady thermoelastic pressure distribution, in that bifurcation in the contact region occurs under conditions which would not lead to bifurcation in the absence of wear. Moreover, wear can lead to local values of pressure which exceed those in the steady state without wear (see Fig. 7). It is suggested that this might be explained by the tendency of wear in combination with thermal expansion to develop a rough surface profile [2].

The pressure and temperature distributions tend eventually to approach a quasi-random state, which is relatively independent of the initial condition. Ideally, we should like to be able to characterize the statistical properties of this stage as functions of the physical parameters of the system. Investigations along these lines are continuing and will be reported later.

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