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THERMOELASTICITY AND CONTACT

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Thermoelastic deformations can have a significant effect on the contact between elastic bodies, particularly in cases where the thermal boundary conditions at the interface are influenced by the contact pressure. In the classical Hertzian problem, the size of the contact area depends on the magnitude and direction of heat flow between the bodies. Idealized thermal boundary conditions can lead to ill-posed steady-state problems, but this difficulty is resolved by assuming a pressure-dependent thermal contact resistance. Steady states of the system can be unstable even when they are unique, in which case the behavior is either oscillatory or involves the steady motion of a contact pressure wave along the interface. Analytical and numerical perturbation methods have been developed to investigate the stability problem. These results find applications in heat transfer processes involving solid-solid contact, including the solidification of castings. In brakes and clutches, the heat generated at the sliding interface causes thermal distortion leading to "frictionally excited thermoelastic instability" or "TEI," in which contact becomes localized in "hot spots" at the interface. Recent results enable us to make good predictions of the conditions under which this occurs.

When two conforming bodies are placed in contact, the contact pressure distribution is sensitive to comparatively small changes in surface profile. Thermoelastic deformations, though generally small, can therefore have a major effect on systems involving contact. Further interesting effects are introduced if the thermal boundary conditions at the interface are influenced by the mechanical contact conditions. The thermal and thermoelastic problems are then coupled through the boundary conditions, and as a consequence the steady-state solution may be nonunique and/or unstable.

Thermoelastic contact problems of this class are found in many applications—one of the most important being sliding systems such as brakes, clutches, and seals, where thermoelastic effects are driven by frictional heat generation that depends on the local pressure [1, 2]. However, coupled problems are also obtained for the

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static conduction of heat across an interface between two thermoelastic bodies because the extent of the contact area influences the heat conduction problem and depends in turn on the thermoelastic distortion. Even if there is full contact between the two bodies, there will generally be a thermal contact resistance at the interface that varies with local contact pressure and we shall show later that this can be a source of thermoelastic contact instability. Conduction across a solid/solid interface forms part of the heat flow path in many heat transfer applications, which can therefore exhibit erratic or nonuniform behavior as a result of such effects. For example, in the nominally one-dimensional solidification of a metal against a plane mold, thermoelastic contact between the partially solidified casting and the mold can become unstable, leading to significantly nonuniform pressure distribution and alloy composition [3, 4].

THE HERTZ PROBLEM

The subject of elastic contact dates back to the classical results of Hertz for the contact of two large bodies with quadratic profiles. A corresponding thermoelastic problem will be obtained if the extremities of the two bodies are maintained at different temperatures T_1, T_2, respectively, so that heat is conducted between them. Most of the heat flow will pass through the contact area, and it is therefore convenient to start with the idealized problem in which no heat flows across the exposed surfaces, while there is perfect thermal contact (continuity of temperature) throughout the contact area. If we restrict attention to the axisymmetric case so that the contact area is a circle, we can take the radius a of this circle as an independent variable and hence solve the heat conduction problem, after which we solve a thermoelastic contact problem for the contact pressure distribution and in particular the contact force P needed to establish a contact area of radius a. The resulting relationship is

$$\frac{P}{Ma^2} = \frac{8(R_1 + R_2)a}{3R_1R_2} + \frac{4}{\pi}\left(\frac{K_1K_2}{K_1 + K_2}\right)(\delta_1 - \delta_2)(T_2 - T_1) \tag{1}$$

[5], where R_1, R_2 are the radii of the contacting bodies,

$$\frac{1}{M} = \frac{(1-\nu_1^2)}{E_1} + \frac{(1-\nu_2^2)}{E_2} \tag{2}$$

$$\delta = \frac{\alpha(1+\nu)}{K} \tag{3}$$

is the thermal distortivity; α, K, E, ν are the coefficient of thermal expansion, thermal conductivity, Young's modulus, and Poisson's ratio; and the suffices refer to bodies 1, 2, respectively.

It is readily verified that Eq. (1) reduces to the classical Hertzian result for the case where $T_2 = T_1$, so there is no heat flow. Another limit of some interest arises if R_1, $R_2 \to \infty$, in which case the second term in Eq. (1) becomes zero and we obtain the limiting contact radius

$$a_0 = \sqrt{\left(\frac{K_1 + K_2}{K_1 K_2}\right) \frac{\pi P}{4M(\delta_1 - \delta_2)(T_2 - T_1)}} \tag{4}$$

In other words, the contact of two plane surfaces will lead to a finite circular contact area that of course is sustained because in this state the thermoelastic distortion causes a “bulge” in body 1 at the contact area.

When $(\delta_1 - \delta_2)(T_2 - T_1) < 0$—that is, when the heat flows into the material with the lower distortivity—Eq. (1) predicts that the thermoelastic distortion will cause an *increase* in the contact radius and hence a decrease in the resistance to heat flow between the bodies afforded by the constriction. However, a closer examination of the solution in this case shows a small region of unacceptable tensile contact tractions near $r = a$. Comninou and Dundurs [6] examined the asymptotic stress and temperature fields near a transition between perfect thermal contact and separation with complete insulation and showed that this transition always leads to such a violation of the unilateral inequalities if the heat flows into the material with the lower distortivity. We conclude that no solution exists to the steady-state problem as posed, for this direction of heat flow.

EXISTENCE AND UNIQUENESS

This paradox is easier to understand in the context of a simple one-dimensional model. Figure 1 shows a thermoelastic rod built into a rigid wall at A and separated from a second rigid wall at B by a small gap g. If the temperature of wall A is now increased, the temperature of the rod will increase until the gap shrinks to zero. We would normally anticipate contact at B for temperatures beyond this critical condition; but as soon as contact occurs, heat will flow along the rod, reducing its mean temperature and hence the thermal expansion. Elemen-

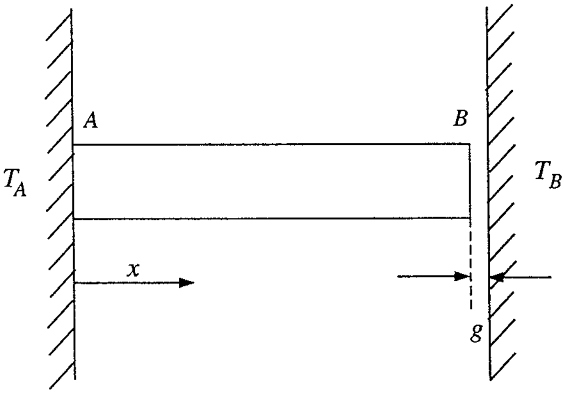

Figure 1. One-dimensional rod model.

tary calculations [7] show that there is a range of temperatures T_A, T_B for which the assumption of perfect thermal contact leads to tensile contact tractions and the assumption of separation leads to interpenetration of material (a negative gap). In other words, there is no steady-state solution satisfying the ideal thermal boundary conditions.

It should be emphasized that the classical existence and uniqueness proofs for heat conduction and thermoelasticity do not apply to this coupled problem. If the temperature field is known there is a unique solution to the corresponding contact problem and if the solution of the contact problem is known there is a unique solution to the heat conduction problem, but neither of these conditions is met because each stage requires the previous solution to be known, as shown in Figure 2. Duvaut [8] showed that an existence theorem can be proved for the coupled problem if a more realistic boundary condition is used in which there is a thermal contact resistance at the interface that varies inversely with the contact pressure.

The thermal resistance at the interface between two contacting solids has been a subject of extensive experimental [9, 10] and theoretical [11, 12] investigations. The resistance can be attributed to two principal sources—the roughness of the contacting solids, which causes intimate contact to be restricted to microscopic "actual contact areas," and the presence of low-conductivity surface films. Both experimental measurements and theoretical predictions are notoriously sensitive to minor changes in conditions or assumptions, but there is general agreement that the resistance is a monotonically decreasing function of contact pressure—a result that can be confirmed very easily by touching a hot (or cold) object, first with light finger pressure and then with a firm grip. Duvaut [8] also proved the uniqueness of the steady-state solution under the condition that this pressure dependence is sufficiently weak, but reported experimental measurements show that this condition is unlikely to be met in practice.

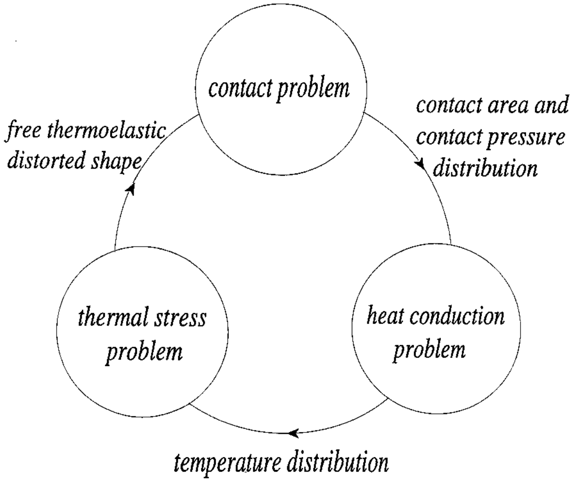

Figure 2. Coupling between the thermal and mechanical problems.

Barber et al. [13] showed that the rod model of Figure 1 always has at least one steady-state solution with the more general boundary condition.

$$Q = \frac{\hat{T}}{R(p, g)} \tag{5}$$

where Q is the heat flux across the interface, $\hat{T}$ is the temperature drop across it, and the contact resistance $R(p, g)$ is a fairly general continuous monotonic function of contact pressure p or gap g. Duvaut's boundary condition is a special case of this form, but the idealized conditions of perfect thermal contact or perfect insulation is not, because it exhibits a discontinuity in resistance at the transition from contact to separation—that is, when $p = 0$ and $g = 0$. A more rigorous proof of existence under these conditions was given by Andrews et al. [14], who also examined the effect of discontinuities in the resistance function.

NONUNIQUENESS AND STABILITY

The rod model of Figure 1 exhibits multiple solutions for certain temperature ranges when $T_B > T_A$. The stability of these solutions can be investigated by superposing an infinitesimal perturbation on the steady state and examining the conditions under which such a perturbation can grow exponentially in time. For example, in the rod model, we assume a temperature field of the form

$$T(x, t) = T_0(x) + T_1(x) e^{bt} \tag{6}$$

where $T_0(x)$ is the temperature distribution in the steady state. Substituting this expression into the heat conduction equation leads to a solution for $T_1(x)$ apart from an arbitrary multiplying constant, after which the solution of the contact problem gives the perturbation in contact pressure or gap. The final stage is to perform a linear perturbation on the contact resistance equation (5) to obtain

$$\Delta\hat{T} = Q_0 R'(p_0) \Delta p + R_0 \Delta Q \tag{7}$$

where $\Delta\hat{T}$, Δp, ΔQ are the perturbations in $\hat{T}$, p, Q, respectively, and Q_0, R_0 are the heat flux and contact resistance in the steady state. Substituting the perturbation quantities into this equation yields a set of homogeneous equations that have a nontrivial solution only for certain eigenvalues of the exponential growth rate b. If we assume that an arbitrary initial perturbation could be expressed in the form of an eigenfunction series—that is, that the set of eigenfunctions is complete on the domain defined by the body—it follows that the general transient solution near the steady state can be written as such a series and hence that the system is unstable if and only if one or more of the eigenvalues b has positive real part.

For the rod model, an odd number of steady-state solutions is always obtained (usually 1 or 3) and they can be shown to be alternately stable and unstable, with unique solutions being invariably stable [13]. However, this simple behavior is a

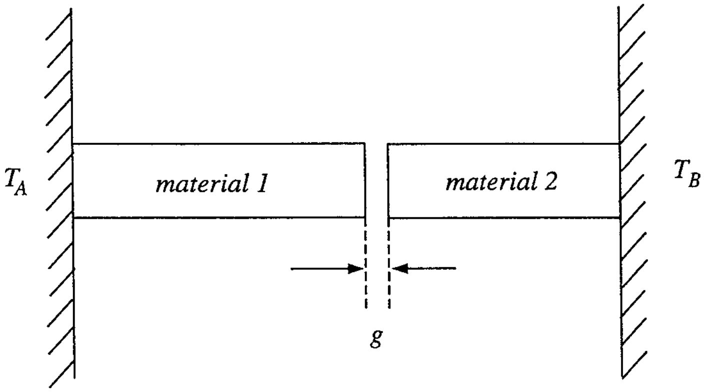

Figure 3. One-dimensional model involving two different materials.

consequence of the fact that only one thermoelastic material is involved. The slightly more complex problem of two contacting rods, shown in Figure 3, also exhibits ranges of unique and multiple steady-state solutions, but there is now no correlation with the stability behavior and indeed conditions can be found where unique steady states are unstable [15]. In such cases, transient numerical simulation of the problem predicts an oscillatory state in which the contact pressure cycles between high and low values or between contact and separation. There is reason to believe that this mechanism may be responsible for reported erratic behavior of systems involving the conduction of heat across an interface between two thermoelastic bodies [16, 17].

Two- and Three-Dimensional Stability Problems

Similar behavior is observed in thermoelastic contact in two and three dimensions, though different techniques are generally needed to solve the corresponding stability problem. If contact occurs on an infinite plane, as in the contact of two half-planes, the eigenmodes must vary sinusoidally in the direction of the interface and the mathematical problem for each Fourier mode is analogous to that for the one-dimensional rod problem [18]. Similar methods, but involving Fourier series, can be used when the problem involves axisymmetric bodies such as the contact of two thin-walled cylinders on a common end face as shown in Figure 4. For some material combinations, the trivial steady-state solution involving uniform pressure is unique but unstable for sufficiently high heat flux q_0. Transient simulations [19] in such cases show that an arbitrarily small initial perturbation is sufficient to precipitate an unstable transition to a state involving one or more contact and separation regions that move along the interface at constant speed, along with the associated thermal and mechanical fields.

Similar analytical methods have been used to investigate the stability of systems involving the contact of layers and thin-walled concentric cylinders. In this case, the dominant eigenmode—i.e., that which goes unstable at the lowest value

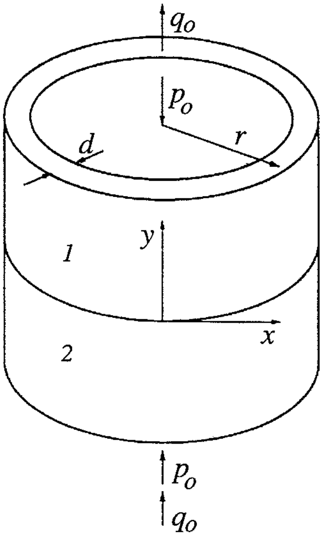

Figure 4. Thermoelastic contact of two thin-walled cylinders on an end face.

of Q_0—generally involves a wavelength that is between two and five times the layer thickness [20].

Numerical Solution of Stability Problems

If the contact area is bounded and not axisymmetric, analytical techniques are of limited value for determining the stability boundary, but in the linear range we can still argue that stability is governed by a perturbation that increases exponentially with time. As before, we therefore postulate that the temperature field is given by

$$T(x, y, z, t) = T_0(x, y, z) + T_1(x, y, z)e^{bt} \tag{8}$$

where T_0 is the steady-state solution and T_1 is the eigenfunction of the temperature perturbation. Corresponding expressions can be written for the stress and displacement components. When Eq. (8) is substituted into the heat conduction equation

$$\nabla^2 T = \frac{1}{k}\frac{\partial T}{\partial t} \tag{9}$$

where k is the thermal diffusivity, we obtain the modified equation

$$\nabla^2 T_1 = \frac{bT_1}{k} \tag{10}$$

for the perturbation term. This is a linear equation in the spatial coordinates x, y, z only with b as a parameter; and it can be discretized by the finite element method [21]. Imposition of the corresponding perturbed boundary conditions leads to a linear eigenvalue problem for the growth rate b.

If the dominant eigenvalue can be assumed to be real, the stability boundary corresponds to the lowest heat flux giving an eigenvalue $b = 0$—in other words, to a condition under which the homogeneous perturbation problem has a nontrivial steady-state solution. This permits a more direct solution of the stability problem in which the stability boundary is obtained directly. However, the restriction to real growth rates is a serious one because the evidence suggests that complex roots are dominant in many problems involving the contact of dissimilar materials [20].

SOLIDIFICATION PROBLEMS

During the casting process, heat is conducted across a solid/solid interface from the partially solidified casting into the mold. The thermal resistance at this interface plays an important role in the evolution of solidification and the development of the final grain structure and residual stress. Ho and Pehlke [22] deduced values of thermal contact resistance from temperature measurements during solidification experiments and found that the resistance generally increases significantly as solidification proceeds. A possible explanation for this phenomenon is suggested by Richmond and Tien [23], who showed that thermoelastic shrinkage of the casting will cause air gaps to form at some locations on the interface. In addition, the cast surface initially conforms with the contacting mold and relative tangential motion due to thermoelastic distortion may reduce the extent of intimate contact between the surfaces [24].

As in the thermoelastic contact of two solids, we should anticipate the possibility of instability associated with the pressure dependence of the contact resistance, and indeed there is ample experimental evidence of waviness in the development of the solidification front in nominally uniform solidification that is probably attributable to this mechanism [3, 25]. This leads to a corresponding nonuniformity in the morphology and concentrations in the solidification of alloys and can even cause remelting in regions where air gaps develop.

Analysis of the stability of solidification presents a new feature in that the unperturbed or zeroth-order solution T_0 in Eq. (8) is itself a function of time, since solidification is inherently a transient process. Thus, the usual methods of perturbation analysis cannot be used and the concept of instability needs redefinition, since an arbitrarily small initial perturbation would not have time to grow to serious proportions during the process. Algebraic solutions have been obtained for a variety of idealized problems, mostly involving pure metals [4, 26, 27]. It should be noted that the zeroth-order process is inherently nonlinear because of the moving solidification boundary, but for small perturbations the perturbation problem is linear, being defined by sets of equations containing the spatial derivatives of the zeroth-order solution. This leads to a set of linear equations with coefficients that vary in a known way in both time and space. The zeroth-order problem can possess fairly general nonlinearities, such as those resulting from the use of temperature-dependent material properties [28]. These effects can be important because properties can vary quite extensively at temperatures near the melting point.

THERMOELASTICITY IN SLIDING CONTACT

If we start with Hertzian contact and then allow one body to slide over the other, two new features enter into the process—frictional heat is generated at the interface and the contact area must now move over at least one of the bodies. The latter effect can be quantified in terms of a Peclet number

$$\mathrm{Pe} = \frac{Va}{k} \tag{11}$$

where a is a representative linear dimension of the contact area and V its velocity relative to the body. Typical Peclet numbers in engineering applications are very large and usually permit a simplification in which the conduction of heat in the plane of the contact area can be neglected. The magnitude of thermoelastic effects in sliding contact problems is governed by the dimensionless material parameter

$$H = \frac{E\alpha}{\rho c(1-\nu)} \tag{12}$$

[29], where ρ is the density and c is the thermal capacity. This parameter is close to unity for a wide range of structural materials, despite considerable disparity in the contributory properties.

Hills and Barber [30] gave an analytical solution for the two-dimensional sliding Hertzian problem, using a thermoelastic Green's function to reduce the problem to the solution of an integral equation with a Bessel function kernel. A remarkable feature of their results was that no steady-state solution could be found in certain ranges of the applied load and sliding speed without violating the unilateral contact constraints. Similar results were demonstrated by Yevtushenko and Ukhanska [31] for the same problem with an interfacial contact resistance, which was not a function of pressure. Jang [32] showed that similar problems arise in the simpler case in which the contacting bodies are replaced by elastic foundations. He developed a numerical algorithm for the transient problem in this case and showed that the contact area tends to break down into a number of smaller regions as sliding progresses. Even more surprising is the fact that this process appears to continue without limit, leading to larger and larger numbers of smaller contact areas.

Frictionally Excited Thermoelastic Instability

One of the most technologically important areas involving thermoelastic contact is that in which the thermal problem is driven by the frictional heat generated during sliding. This coupled process is susceptible to *thermoelastic instability* (TEI) if the sliding speed is sufficiently high. If one of the two materials is a rigid nonconductor, the critical sliding speed V_{cr} for a problem of any two-dimensional (plane strain) geometry can be written [33] in the form

$$V_{cr} = \frac{CK(1-v)}{fE\alpha a} \tag{13}$$

where C is a dimensionless shape factor, a is a representative dimension of the body, and f is the coefficient of friction. Above the critical speed, a nominally uniform pressure distribution is unstable, giving way to localization of load and heat generation and hence to hot spots at the sliding interface [34]. These in turn can cause material damage and wear and are also a source of undesirable frictional vibrations [35]. For the system to be stable, the critical speed must be greater than the operating speed so, for example, it is generally desirable to use materials with high conductivity and low elastic modulus and expansion coefficient. Similar problems arise in the sliding contact of electrical brushes, where they are complicated by electrical resistance heating [36].

Burton et al. [37] developed the perturbation method, described in the "Non-Uniqueness and Stability" section of this paper, to investigate the stability of contact between two sliding half-planes. This method has since been used for other geometries, including a solution by Lee and Barber [1] for a layer sliding between two half-planes, as shown in Figure 5. Lee's solution shows that eigenmodes have the sinusoidal form

$$T(x, y, t) = f(y)\cos\{m(x - ct)\} \tag{14}$$

and the critical speed is a function of wavenumber m, as shown in Figure 6. For most material combinations, the dominant eigenmode—i.e., that which goes unstable at the lowest critical speed—is an antisymmetric mode with a wavelength related to the layer thickness. Antisymmetric eigenmodes correspond to cases in which hot spots alternate on the two sides of the layer. These results have been

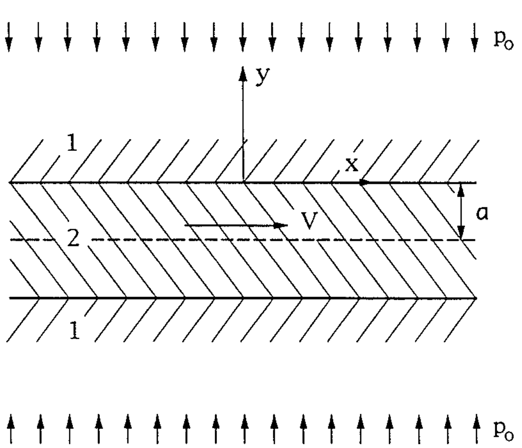

Figure 5. A layer (2) sliding between two half-planes (1).

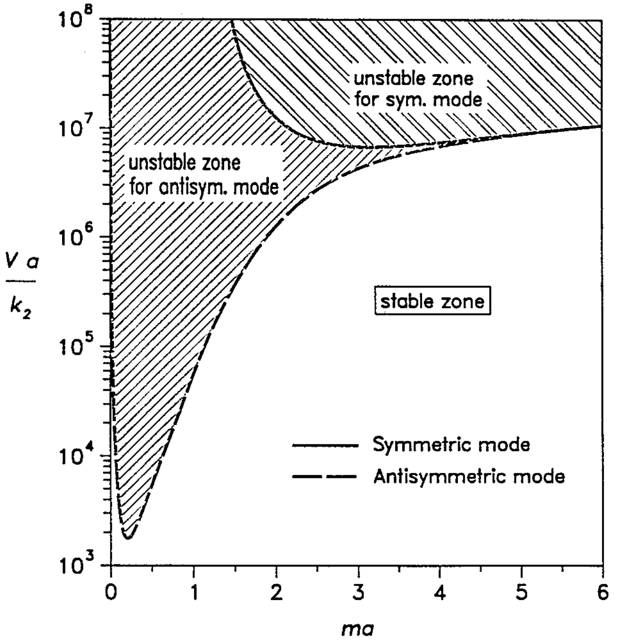

Figure 6. Stability chart for the system of Figure 5 with typical automotive brake materials.

shown to give quite good predictions of the critical speed and the number and spacing of hot spots for automotive disk brakes and clutches [35].

These analytical methods show that the unstable perturbation migrates with respect to both bodies, unless one body is a nonconductor, in which case the perturbation is stationary in the conductor. If the materials have very dissimilar conductivities, the migration speed in the good conductor is small and the expansion of this body dominates the growth of the perturbation, since little time is available for the development of a deformed profile if the migration speed is large. As a consequence, critical speeds are generally increased when the conductivities of the two materials are comparable. For the special case of similar materials, the perturbation would be expected to move at the mean speed of the sliding components [37], but experimental results show that even in this case, the system tends to select an unstable mode in which hot spots are stationary in one of the two bodies [38].

Du et al. [39] have implemented Burton's perturbation method numerically using the method described in the "Numerical Solution of Stability Problems" in this paper. Yi et al. [40] applied this method to the automotive disk brake geometry and showed that focal hot spots tend to be produced in the disk, with a spacing close to that predicted by Lee's layer solution. A more direct approach to the investigation of TEI is to use a numerical method to simulate the behavior of the coupled transient thermoelastic contact problem in time [2, 41]. A recent simulation of Lee's layer geometry [42] showed that the migration of the perturbation ceases when separation occurs, leading eventually to hot spots that are stationary in the better conductor. This result agrees with the limited amount of reported experimental data. An interesting result is that if heat transfer across the gap in separation regions is introduced into the model, migration continues and hot spots continue to migrate even in the steady state.

The simulation method is extremely computer-intensive, but it has the advantage that it is readily adapted to practical loading cycles, which is of importance in the application to transmission clutches, which experience intense periods of operation with rapidly varying sliding speed. Analytical methods of treating some idealized problems with variable sliding speed are discussed by Olesiak [43] and Yevtushenko and Chapovska [44].

CONCLUSIONS

This brief review demonstrates that the thermomechanical coupling associated with thermal boundary conditions that depend on contact pressure leads to an extraordinarily rich variety of physical phenomena, none of which can occur in the absence of coupling. Many of these effects are still imperfectly understood and much research remains to be done.

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