STEADY-STATE THERMAL STRESSES IN AN ELASTIC SOLID CONTAINING AN INSULATED PENNY-SHAPED CRACK

J. R. BARBER *Department of Mechanical Engineering, University of Newcastle upon Tyne*

A solution is given for the steady-state thermal stress and displacement field in an infinite elastic solid containing an insulated penny-shaped crack. The problem is reduced to a mixed-boundary-value problem for the half-space, making use of Green's isothermal solution for the thick elastic plate in complex harmonic potentials and a particular thermoelastic solution due to Williams. In the axisymmetric case, the complex potential reduces to the real harmonic function used by Shail in his solution for the external crack.

To illustrate the use of the method in both axisymmetric and non-axisymmetric problems, complete solutions are given for (1) a uniform heat flow and (2) a linearly varying heat flow disturbed by an insulated penny-shaped crack.

1 INTRODUCTION

The thermoelastic-stress distribution in a large solid containing an insulated penny-shaped crack can be obtained by superposing

(1) the solution for the corresponding *unflawed* solid;
(2) an *isothermal* solution in which tractions are applied to the crack surfaces equal and opposite to those transmitted through the same area in (1), whilst the extremities of the body are stress-free;
(3) a thermoelastic solution in which a heat flux is imposed across the crack surfaces equal and opposite to that transmitted through the corresponding area in (1), whilst all the surfaces of the body (including the crack) are stress-free.

If the body is large in comparison with the diameter of the crack, it will be acceptable to treat it as infinite in obtaining the solutions (2) and (3).

The solution of problem (3) for an arbitrary temperature field can be considered as the sum of two components: one symmetric and the other anti-symmetric with respect to the crack plane. In physical terms, the symmetric component gives the effect of a distributed heat source at the crack surface, whilst the anti-symmetric component corresponds to the case in which a heat flux across the crack plane is disturbed by the presence of an insulated crack. There are comparatively few mechanisms by which heat can be generated at an enclosed crack surface and hence the anti-symmetric solution would seem to be of the greater practical interest. Nonetheless, previous solutions have been largely restricted to the symmetric case (1) (2)*, though a particular anti-symmetric problem – that in which the undisturbed heat flow would have been uniform – has been treated by Florence and Goodier (3).

This paper gives a general solution to the anti-symmetric problem in which the temperature field has reached a steady state. The method used is broadly similar to that developed by Shail (4) and Rubenfeld (5) for the analogous external-crack problem except that use is made of features from papers by Green (6) (7) and Copson (8) to simplify the statement and solution of the resulting mixed-boundary-valve problems.

2 GENERAL SOLUTION

A suitable solution to the equations of thermoelastic equilibrium can be obtained by combining the particular steady-state thermoelastic solution of Williams (2) with an isothermal solution due to Green (6). In this paper a cartesian co-ordinate system x, y, z is used and the complex variable $x + iy$ and its conjugate are denoted by ζ, $\bar{\zeta}$ respectively. In general, a bar placed over a function denotes its complex conjugate.

The displacement components (u_x, u_y, u_z) and stress components (σ_{xx}, σ_{xy}, etc.) can be expressed in the form

$$\left.\begin{aligned} D \equiv u_x + iu_y &= 4\frac{\partial F}{\partial \bar{\zeta}} \\ u_z &= \frac{\partial G}{\partial z} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\left.\begin{aligned} \Theta \equiv \sigma_{xx} + \sigma_{yy} &= -\frac{\mu}{(1-\nu)} \left\{\frac{\partial^2 H}{\partial z^2} - 2\nu\frac{\partial^2 G}{\partial z^2} + 2\alpha(1+\nu)T\right\} \\ \Phi \equiv \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} &= 16\mu\frac{\partial^2 F}{\partial \bar{\zeta}^2} \\ \Psi \equiv \sigma_{xz} + i\sigma_{yz} &= 2\mu\frac{\partial^2 (2F+G)}{\partial \bar{\zeta}\partial z} \\ \sigma_{zz} &= -\mu\nabla_1^2 H \end{aligned}\right\} \quad (2)$$

where $\nabla_1^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \equiv 4\dfrac{\partial^2}{\partial \zeta \partial \bar{\zeta}}$

The MS. of this paper was received at the Institution of Mechanical Engineers on 6th September 1974 and accepted for publication on 29th October 1974. 32

---

*References are given in the Appendix.

α, μ, ν are the coefficients of thermal expansion, modulus of rigidity and Poisson's ratio respectively for the material; T is temperature; and F, G, H are three functions of ζ, $\bar{\zeta}$, z satisfying the equations

$$\left.\begin{aligned} 2(1-2\nu)\nabla^2 F &= (1-2\nu)\nabla^2 G \\ &= 2\alpha(1+\nu)T - \nabla_1^2(F+\bar{F}) - \frac{\partial^2 G}{\partial z^2} \\ H &= F+\bar{F}+G \end{aligned}\right\} \quad . \quad . \quad (3)$$

A solution of equations (3) which is sufficiently general for the present problem is

$$\left.\begin{aligned} 2F &= \phi - i\psi + \frac{z}{2(1-\nu)}\frac{\partial}{\partial z}(\phi-\omega) \\ G &= -\phi + \frac{z}{2(1-\nu)}\frac{\partial}{\partial z}(\phi-\omega) \\ H &= \frac{z}{(1-\nu)}\frac{\partial}{\partial z}(\phi-\omega) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ϕ, ψ, ω are real harmonic functions and

$$\frac{\partial^2\omega}{\partial z^2} = -\alpha(1+\nu)T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Apart from certain constant multiplying factors, these three stress functions correspond to those used by Rubenfeld (5) for the external-crack problem. However, the complex-variable formulation outlined above permits a rather more compact statement of the solution and greatly facilitates co-ordinate transformations.

The components of displacement and stress in cylindrical polar co-ordinates (r, θ, z) are related to those given above by the equations

$$\left.\begin{aligned} D' &\equiv u_r + iu_\theta = e^{-i\theta}D \\ \Theta' &\equiv \sigma_{rr} + \sigma_{\theta\theta} = \Theta \\ \Phi' &\equiv \sigma_{rr} - \sigma_{\theta\theta} + 2i\sigma_{r\theta} = e^{-2i\theta}\Phi \\ \Psi' &\equiv \sigma_{rz} + i\sigma_{\theta z} = e^{-i\theta}\Psi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

3 THE ANTI-SYMMETRIC CRACK PROBLEM

3.1 Boundary conditions

Let us suppose that the crack extends over the region $a \geqslant r \geqslant 0$, $z = 0$ in cylindrical polar co-ordinates. Since the problem is anti-symmetric with respect to the plane $z = 0$, we must have

$$\sigma_{zz}, D, T = 0; r > a, z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

whilst on both crack surfaces we have

$$\sigma_{zz}, \Psi = 0; \frac{\partial T}{\partial z} = -\frac{Q}{K}; a \geqslant r \geqslant 0, z = 0 \quad . \quad . \quad . \quad . \quad (8)$$

where Q is a prescribed function of ζ, $\bar{\zeta}$ and K is the thermal conductivity of the material.

Substituting for F, G, H from equation (4) into equation (2), we find that the conditions on σ_{zz} are satisfied identically whilst the other boundary conditions require

$$\left.\begin{aligned} &\frac{\partial}{\partial\bar{\zeta}}(\phi - i\psi) = 0 \\ &\frac{\partial^2\omega}{\partial z^2} = 0 \end{aligned}\right\} r > a, z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\left.\begin{aligned} &\frac{\partial}{\partial\bar{\zeta}}\left\{\frac{\partial\phi}{\partial z} - \frac{\partial\omega}{\partial z} - i(1-\nu)\frac{\partial\psi}{\partial z}\right\} = 0 \\ &\frac{\partial^3\omega}{\partial z^3} = \frac{\alpha(1+\nu)Q}{K} \end{aligned}\right\} a \geqslant r \geqslant 0, z = 0 \quad . \quad . \quad . \quad (10)$$

Let us suppose that the heat flux Q on the crack face can be expressed in the form

$$Q = q(r)\cos(n\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where q is a function of r only and n is a positive integer or zero. More general functions can be expressed as a sum of terms of this kind.

3.2 Solution for $\partial\omega/\partial z$

The boundary-value problem for $\partial\omega/\partial z$ defined by equations (9)–(11) is of a well known type and can be solved by Green's method (7). We write

$$\frac{\partial\omega}{\partial z}(r, \theta, 0) = p(r)\cos(n\theta); a \geqslant r \geqslant 0 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and, since ω is harmonic, it follows that

$$\begin{aligned} \nabla_1^2\{p(r)\cos(n\theta)\} &= -\frac{\partial^3\omega}{\partial z^3}(r, \theta, z); a \geqslant r \geqslant 0 \\ &= -\frac{\alpha(1+\nu)q(r)\cos(n\theta)}{K} \end{aligned} \quad . \quad . \quad (13)$$

Hence

$$r^{n-1}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r^{1-2n}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^n p\right)\right\} = -\frac{\alpha(1+\nu)q}{K} \quad . \quad . \quad (14)$$

In terms of this function, we then have

$$\frac{\partial^2\omega}{\partial z^2}(r, \theta, 0) = \frac{2r^{n-1}\cos(n\theta)}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{t^{1-2n}}{\sqrt{t^2-r^2}}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^t \frac{s^{n+1}p(s)\mathrm{d}s}{\sqrt{t^2-s^2}}\right\}\mathrm{d}t; a \geqslant r \geqslant 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

from equations 2:17 and 2:18 of (7). The arbitrary constants in the solution of equation (14) are found from conditions of continuity (a) at the origin and (b) at $r \rightarrow a^-$ (where temperature and hence $\frac{\partial^2\omega}{\partial z^2}$ must tend to zero).

3.3 Solution for ϕ, ψ

Knowing $\partial\omega/\partial z$ on the crack surface, we can now substitute into equations (9) and (10) to set up the boundary-value problem for ϕ, ψ. From equation (9) it is clear that the complex harmonic function ($\phi - i\psi$) must be independent of $\bar{\zeta}$ and hence a function of ζ only on $z = 0$, $r > a$.

Since we are considering terms of the form of equation (11), it follows that

$$\left.\begin{aligned}\phi - i\psi &= c_1\zeta^{-n}; n \neq 0, z = 0, r > a\\ &= 0; n = 0, z = 0, r > a\end{aligned}\right\} \quad \ldots\ldots (16)$$

where c_1 is an arbitrary constant.

Applying a similar argument to equation (10) we obtain

$$\frac{\partial\phi}{\partial z} - \frac{\partial\omega}{\partial z} - i(1-\nu)\frac{\partial\psi}{\partial z} = c_2\zeta^n; z = 0, a \geqslant r \geqslant 0 \quad \ldots (17)$$

Non-axisymmetric case

Let us first restrict our attention to the more interesting case in which $n \neq 0$. In view of equation (16) we shall write

$$\phi - i\psi = \left\{\frac{\partial f_1}{\partial z} + \frac{\partial f_2}{\partial z}\right\} e^{-in\theta} + \frac{\partial g}{\partial z}\cos(n\theta) \quad \ldots (18)$$

where f_1, f_2, g are real functions of r, z only and

$$\frac{\partial f_1}{\partial z}, \frac{\partial g}{\partial z} = 0; z = 0, r > a \quad \ldots\ldots\ldots (19)$$

$$\frac{\partial f_2}{\partial z} = c_1 r^{-n}; z = 0, \textit{all } r > 0 \quad \ldots\ldots\ldots (20)$$

Substituting for ϕ, ψ, ω from equations (12) and (18) into equation (17), we also have the conditions

$$\frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_2}{\partial z^2} = -c_2 r^n; z = 0, a \geqslant r \geqslant 0 \quad \ldots\ldots (21)$$

$$\frac{\partial^2 g}{\partial z^2} = p(r) + \frac{(2-\nu)c_2 r^n}{(1-\nu)}; z = 0, a \geqslant r \geqslant 0 \quad \ldots (22)$$

The boundary-value problems for f_1, g defined by equations (19) (21) (22) are similar to the problem already solved for ω, except that the requirements of continuity as $r \to a^-$ are more stringent. Thus, for continuity of displacement we must have

$$\underset{r \to a^-}{Lt}(D) = 0$$

which will be satisfied if

$$\underset{r \to a^-}{Lt}\left(\frac{\partial f_1}{\partial z}, \frac{\partial g}{\partial z}, \frac{\partial^2 f_1}{\partial r \partial z}, \frac{\partial^2 g}{\partial r \partial z}\right) = 0 \quad \ldots\ldots (23)$$

from equations (1) (4) (18).

If $h(r)$ is a particular integral of the equation

$$r^{n-1}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r^{1-2n}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^n h\right)\right\} = -p(r) \quad \ldots (24)$$

it follows that

$$g(r,0) = h(r) - \frac{(2-\nu)c_2 r^{n+2}}{4(n+1)(1-\nu)} + c_3 r^n; a \geqslant r \geqslant 0$$

and hence

$$\frac{\partial g}{\partial z}(r,0) = \frac{2r^{n-1}}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{tV(t)\mathrm{d}t}{\sqrt{t^2-r^2}}; a \geqslant r \geqslant 0 \quad \ldots (25)$$

where we adopt the notation

$$U, V, W(t) = t^{-2n}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{s^{n+1}\{f_1, g, h(s)\}\mathrm{d}s}{\sqrt{t^2-s^2}} \quad \ldots (26)$$

(cf. equation (15)).

The continuity conditions (23) can only be satisfied if

$$\underset{t \to a^-}{Lt}(V(t), V'(t)) = 0 \quad \ldots\ldots\ldots\ldots (27)$$

in which case, equation (25) gives

$$\frac{\partial g}{\partial z}(r,0) = \frac{2r^n}{\pi}\int_r^a \frac{V'(t)\mathrm{d}t}{\sqrt{t^2-r^2}} \quad \ldots\ldots\ldots (28)$$

These conditions (27) enable us to find the two arbitrary constants c_2, c_3, and hence $V(t)$, $V'(t)$. Using the notation of equation (26) we find

$$V'(t) = W'(t) - t/a\, W'(a) \quad \ldots\ldots\ldots (29)$$

$$c_2 = \frac{(2n+1)!!\,(1-\nu)W'(a)}{(2n)!!\,(2-\nu)a} \quad \ldots\ldots\ldots (30)$$

where $(2n)!! = 2.4.6.\ldots(2n)$ and $(2n+1)!! = 1.3.5.\ldots(2n+1)$.

The corresponding solution for $f_1(r, 0)$ can now be completed, making use of this value of c_2.

We have

$$f_1(r,0) + f_2(r,0) = \frac{c_2 r^{n+2}}{4(n+1)} + c_4 r^n; a \geqslant r \geqslant 0 \quad \ldots (31)$$

from equation (21). The boundary conditions on f_2 are unmixed and hence we can find $f_2(r, 0)$ directly by integration, i.e.

$$f_2(r,0) = \frac{-2}{\pi r^n}\int_0^r \frac{t^{2n}}{\sqrt{r^2-t^2}}\left\{\int_t^\infty \frac{c_1 s^{1-2n}\mathrm{d}s}{\sqrt{s^2-t^2}}\right\}\mathrm{d}t;$$
$$r > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

(see equation (5) of Copson (8))

$$= -\frac{(2n-3)!!c_1}{(2n-2)!!r^{n-1}} \quad \ldots\ldots\ldots\ldots (33)$$

We have

$$f_1(r,0) = \frac{(2n+1)!!\,(1-\nu)W'(a)r^n}{2(2n+2)!!\,(2-\nu)a} + c_4 r^n +$$
$$+ \frac{(2n-3)!!c_1}{(2n-2)!!r^{n-1}}; a \geqslant r \geqslant 0 \quad \ldots\ldots\ldots (34)$$

from equations (30) (31) (33) and

$$\frac{\partial f_1}{\partial z}(r,0) = \frac{2r^n}{\pi}\int_r^a \frac{U'(t)\mathrm{d}t}{\sqrt{t^2-r^2}}; a \geqslant r \geqslant 0 \quad \ldots\ldots (35)$$

(cf. equation (28)).

Substituting for $f_1(s, 0)$ from equation (34) into equation (26) and applying the two continuity conditions

(23) relating to f_1 to evaluate c_1, c_4, we find

$$U'(t) = \frac{W'(a)}{(2-\nu)} \left\{ t/a - (a/t)^{2n} \right\} \quad \ldots \ldots \quad (36)$$

$$c_1 = \frac{2(2n-2)!!\, a^{2n}\, W'(a)}{\pi(2n-1)!!\,(2-\nu)} \quad \ldots \ldots \quad (37)$$

The functions which appear in equation (18) are now defined throughout the plane $z = 0$ in unmixed form and the solution for ϕ, ψ can be completed by standard methods.

Axisymmetric case

If we apply the same method to the case in which $n = 0$, we find that the right-hand side of equation (18) reduces to the sum of three real axisymmetric functions and no loss of generality is involved in writing

$$\phi - i\psi = \frac{\partial g}{\partial z} \quad \ldots \ldots \quad (38)$$

where

$$\frac{\partial g}{\partial z} = 0; z = 0, r > a \quad \ldots \ldots \quad (39)$$

$$\frac{\partial^2 g}{\partial z^2} = p(r) + c; z = 0, a \geqslant r \geqslant 0 \quad \ldots \ldots \quad (40)$$

The solution then proceeds as in the previous section to give

$$\frac{\partial g}{\partial z}(r, 0) = \frac{2}{\pi} \int_r^a \frac{\{W'(t) - t/a\, W'(a)\}\, \mathrm{d}t}{\sqrt{t^2 - a^2}} \quad \ldots \ldots \quad (41)$$

where $W(t)$ is defined by equation (25) with $n = 0$.

From equation (38), it follows that the function ψ is everywhere zero when the system is axisymmetric. In this case, the solution derived in section 2 reduces to that used by Shail (4) in his treatment of the axisymmetric thermal stress near an external crack in an infinite solid.

4 EXAMPLES

To illustrate the use of the method in both axisymmetric and non-axisymmetric problems, solutions are given for (1) a uniform heat flow and (2) a linearly varying heat flow disturbed by an insulated penny-shaped crack.

4.1 Uniform heat flow

The axisymmetric case of uniform heat flow has been solved by Florence and Goodier (3) using a different method. Substituting $Q = q_0$ (constant) into equations (11) and (14) and integrating, we find

$$p(r) = -\frac{\alpha(1+\nu) q_0 r^2}{4K} + A \log r + B \quad \ldots \ldots \quad (42)$$

where A and B are arbitrary constants, the first of which must be zero to ensure continuity at the origin. Substituting in equation (15) and applying the condition that $\partial^2\omega/\partial z^2$ is bounded as $r \to a^-$, we find

$$B = \frac{\alpha(1+\nu) q_0 a^2}{2K} \quad \ldots \ldots \quad (43)$$

$$\frac{\partial^2 \omega}{\partial z^2}(r, \theta, 0) = -\frac{2\alpha(1+\nu) q_0 \sqrt{a^2 - r^2}}{\pi K}\; ; a \geqslant r \geqslant 0 \quad . \quad (44)$$

A suitable particular integral of equation (24) with $n = 0$ is

$$h(r) = -\frac{\alpha(1+\nu) q_0}{64K}(8a^2 r^2 - r^4) \quad \ldots \ldots \quad (45)$$

and hence, from equations (26) and (41), we have

$$\frac{\partial g}{\partial z}(r, 0) = \phi(r, \theta, 0) = -\frac{2\alpha(1+\nu) q_0 (a^2 - r^2)^{\frac{3}{2}}}{9\pi K}\; ;$$
$$a \geqslant r \geqslant 0 \quad \ldots \ldots \quad (46)$$

We now have the boundary conditions on ϕ, ω in unmixed form and values at an internal point of the solid can be obtained by integration. We notice, however, that

$$\phi(r, \theta, z) = \frac{\alpha(1+\nu) q_0}{12\pi K} \left\{ z(3r^2 - 2a^2 - 2z^2) i \log\left(\frac{R_2 + z + ia}{R_1 + z - ia}\right) + \frac{11i}{3}(R_2^3 - R_1^3) + 5i(a^2 - r^2)(R_2 - R_1) + 9az(R_2 + R_1) \right\} \quad \ldots \ldots \quad (47)$$

$$\frac{\partial \omega}{\partial z}(r, \theta, z) = \frac{\alpha(1+\nu) q_0}{4\pi K} \left\{ (r^2 - 2z^2 - 2a^2) i \log\left(\frac{R_2 + z + ia}{R_1 + z - ia}\right) + 3iz(R_2 - R_1) + a(R_2 + R_1) \right\} . \quad (48)$$

are real axisymmetric harmonic functions which satisfy the boundary conditions of equations (9) (19) (46) and suitable continuity conditions, where

$$R_2 = \bar{R}_1 = \rho e^{\frac{1}{2}\nu}, (\rho \geqslant 0) \quad \ldots \ldots \quad (49)$$

$$\left.\begin{aligned} \rho^2 \cos(\nu) &= r^2 + z^2 - a^2 \\ \rho^2 \sin(\nu) &= 2za \end{aligned}\right\} \pi \geqslant \nu \geqslant 0 \quad \ldots \ldots \quad (50)$$

For more details of this method of developing harmonic functions to satisfy given discontinuous boundary conditions on a surface plane, the reader is referred to Green (7) and Love (9). A more direct integral formulation of the method applied to axisymmetric functions is given by Green and Zerna (10).

On the boundary, $z = 0$, we have

$$\left.\begin{aligned} R_2 &= R_1 = \sqrt{r^2 - a^2}; r > a \\ R_2 &= -R_1 = i\sqrt{a^2 - r^2}; a \geqslant r \geqslant 0 \end{aligned}\right\} \quad \ldots \ldots \quad (51)$$

The stress and displacement components throughout the solid can now be readily found by substituting from equations (47) and (48) into the general solution (equations (1) (2) (4)). The values obtained on the surface plane are given in Table 1.

Table 1. Values obtained on surface plane

	$a \geqslant r \geqslant 0$	$r > a$
$\dfrac{3\pi K D'}{2\alpha(1+\nu)q_0}$	$r\sqrt{a^2-r^2}$	0
$\dfrac{2\pi(1-\nu)Ku_z}{\alpha(1+\nu)q_0}$	$\left\{(1-\nu)r^2-\dfrac{2}{3}(2-\nu)a^2\right\}\dfrac{\pi}{2}$	$\left\{(1-\nu)r^2-\dfrac{2}{3}(2-\nu)a^2\right\}\sin^{-1}a/r$ $-(1-\nu)a\sqrt{r^2-a^2}$
$\dfrac{\pi K(1-\nu)\Theta'}{4\mu\alpha(1+\nu)q_0}$	$-(1-\nu)\sqrt{a^2-r^2}-\dfrac{(1+\nu)a^2}{3\sqrt{a^2-r^2}}$	0
$\dfrac{3\pi K\Phi'}{4\mu\alpha(1+\nu)q_0}$	$\sqrt{a^2-r^2}-\dfrac{a^2}{\sqrt{a^2-r^2}}$	0
$\dfrac{3\pi K\Psi'}{2\mu\alpha(1+\nu)q_0}$	0	$\dfrac{a^3}{r\sqrt{r^2-a^2}}$

These results agree with those of Florence and Goodier (3) except that (1) the external expression for u_z is not obtained by these authors in closed form and (2) there appear to be misprints in their equations (27) and (28).

4.2 Linearly varying heat flow

Let us now consider the non-axisymmetric case in which the heat flux is given by

$$Q = q_1 r\cos\theta ; a \geqslant r \geqslant 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (52)$$

where q_1 is a constant.

From equations (14) and (15), applying appropriate continuity conditions, we find

$$p(r) = \frac{\alpha(1+\nu)q_1 r(4a^2-3r^2)}{24K} \quad . \; . \; . \; . \; . \; . \; . \; . \; (53)$$

$$\frac{\partial^2\omega}{\partial z^2}(r,\theta,0) = -\frac{4\alpha(1+\nu)q_1 r\sqrt{a^2-r^2}\cos\theta}{3\pi K} ; a \geqslant r \geqslant 0 \quad . \; . \; . \; (54)$$

The real harmonic function

$$\frac{\partial\omega}{\partial z}(r,\theta,z) = -\frac{\alpha(1+\nu)q_1\cos\theta}{24\pi K r}\Bigg\{ ir^2(12z^2+4a^2-3r^2)$$
$$\log\left(\frac{R_2+z+ia}{R_1+z-ia}\right) - i(15r^2+8a^2)\{(z+ia)R_2-(z-ia)R_1\}$$
$$-24ar^2(R_2+R_1)+2i\{(z+ia)R_2^3-(z-ia)R_1^3\}$$
$$+8a(R_2^3+R_1^3)\Bigg\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (55)$$

satisfies equations (9) and (54) on the plane $z = 0$ and appropriate conditions as $R \to \infty$ and therefore gives the complete solution for $\partial\omega/\partial z$.

A particular integral of equation (24) is

$$h(r) = -\frac{\alpha(1+\nu)q_1(4a^2r^3-r^5)}{192K} \quad . \; . \; . \; . \; . \; . \; . \; (56)$$

whence

$$W(t) = \frac{\alpha(1+\nu)q_1(3t^3-5a^2t)}{45K} \quad . \; . \; . \; . \; . \; . \; . \; . \; (57)$$

from equation (26). Substituting this expression into equations (29) and (28) we obtain

$$\frac{\partial g}{\partial z}(r,0) = -\frac{4\alpha(1+\nu)q_1 r(a^2-r^2)^{\frac{3}{2}}}{45\pi K} ; a \geqslant r \geqslant 0 \quad . \; . \; (58)$$

whilst from equations (36) (37) (35) we have

$$\frac{\partial f_1}{\partial z}(r,0) = \frac{4\alpha(1+\nu)q_1 a^2(a^2-r^2)^{\frac{3}{2}}}{45\pi K(2-\nu)r} ; a \geqslant r \geqslant 0 \quad . \; . \; (59)$$

$$\frac{\partial f_2}{\partial z}(r,0) = -\frac{4\alpha(1+\nu)q_1 a^5}{45\pi K(2-\nu)r} ; r > 0 \quad . \; . \; . \; . \; . \; (60)$$

Combining these functions in accordance with equation (18), we obtain

$$\left.\begin{aligned} \phi - i\psi &= \frac{4\alpha(1+\nu)q_1}{45\pi K}\left\{\frac{a^2\{(a^2-r^2)^{\frac{3}{2}}-a^3\}e^{-i\theta}}{(2-\nu)r}\right. \\ &\left.\quad -r(a^2-r^2)^{\frac{3}{2}}\cos\theta\right\} ; z=0, a \geqslant r \geqslant 0 \\ &= -\frac{4\alpha(1+\nu)q_1 a^5 e^{-i\theta}}{45\pi K(2-\nu)r} ; z = 0, r > a \end{aligned}\right\} \quad . \; . \; . \; . \; (61)$$

The values of this function within the solid can be expressed as

$$\phi - i\psi = \frac{\alpha(1+\nu)q_1 a^2 e^{-i\theta}}{15\pi K(2-\nu)r}\Bigg\{ ir^2 z\log\left(\frac{R_2+z+ia}{R_1+z-ia}\right)$$
$$+\frac{i}{3}(R_2^3-R_1^3)-ir^2(R_2-R_1)+ar(z+ia)R_2$$
$$+ar(z-ia)R_1+\frac{4a^3}{3}$$
$$+\frac{\alpha(1+\nu)q_1\cos\theta}{120\pi K r}\Bigg\{ izr^2(15r^2-20z^2-12a^2)$$
$$\log\left(\frac{R_2+z+ia}{R_1+z-ia}\right)+i(8a^2+35r^2)(a^2-r^2)(R_2-R_1)$$
$$+\frac{i}{3}(18a^2+95r^2)(R_2^3-R_1^3)-2i(R_2^5-R_1^5)$$
$$+az(8a^2+75r^2)(R_2+R_1)-10az(R_2^3+R_1^3)\Bigg\} \quad . \; . \; (62)$$

The solution can now be completed by substituting for ϕ, ψ, $\partial\omega/\partial z$ from equations (55) and (62) into the general solution of section 2.

APPENDIX

REFERENCES

(1) OLESIAK, Z. and SNEDDON, I. N. 'The distribution of thermal stress in an infinite elastic solid containing a penny-shaped crack', *Archs ration. Mech. Analysis* 1960 4, 238.

(2) WILLIAMS, W. E. 'A solution of the steady-state thermoelastic equations', *Z. angew. Math. Phys.* 1961 **12**, 452.

(3) FLORENCE, A. L. and GOODIER, J. N. 'The linear thermoelastic problem of uniform heat flow disturbed by a penny-shaped insulated crack', *Int. J. Engng Sci.* 1963 **1**, 533.

(4) SHAIL, R. 'Some steady state thermoelastic stress distributions in the vicinity of an external crack in an infinite solid', ibid. 1968 6, 685.

(5) RUBENFELD, L. 'Non-axisymmetric thermoelastic stress distribution in a solid containing an external crack', ibid. 1970 8, 499.

(6) GREEN, A. E. 'The elastic equilibrium of isotropic plates and cylinders', *Proc. R. Soc.* Series A, 1949 **195**, 533.

(7) *idem.* 'On Boussinerq's problem and penny-shaped cracks', *Proc. Cambridge Phil. Soc. math. phys. Sci.* 1949 **45**, 251.

(8) COPSON, E. T. 'On the problem of the electrified disc', *Proc. Edinb. math. Soc.* 1947 8, 14.

(9) LOVE, A. E. H. 'Boussinerq's problem for a rigid cone', *Q. Jl. Math.* 1939 **10**, 161.

(10) GREEN, A. E. and ZERNA, W. *Theoretical elasticity* 1954 172–8 (Clarendon Press, Oxford).