# THE ROLLING CONTACT OF MISALIGNED ELASTIC CYLINDERS

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A solution is given for the tangential tractions developed between two rolling cylinders of identical elastic materials, whose axes are slightly misaligned. A critical misalignment angle is found, above which slip occurs throughout the contact area. For smaller angles, the contact area contains regions of adhesion and microslip. Results are given for the extent of these regions and for the axial force generated due to the misalignment.

## 1 INTRODUCTION

In a recent paper, Engel and Adams (1)‡ describe a series of experiments and an approximate analysis on the wear due to misalignment of a pair of cylindrical rollers transmitting a normal force but no torque. From an engineering point of view, their most striking conclusion is the relatively small misalignment angle ($<0.2°$) at which significant microslip and hence wear can occur, since perfect alignment cannot be guaranteed in any practical system.

In this paper, a more exact analysis of the contact problem is given, taking account of the extent and influence of microslip and of the two-dimensional nature of the traction distribution.

## 2 STATEMENT OF THE PROBLEM

The problem to be considered is the rolling contact of two cylinders of identical elastic materials, but different radii, $R_1$ and $R_2$, whose axes are misaligned by a small angle, $\phi$.

The misalignment has two principal effects: the contact area will become an ellipse rather than a strip, and the rollers will tend to run off along the axis, if unconstrained. It follows that an axial force will be generated if the rollers are in fixed bearings and this force must be distributed over the contact area as a tangential traction. In general, we might expect the contact area to contain regions of adhesion and regions of microslip.

It will be assumed throughout that there is no net force in the rolling direction—i.e., that no torque is transmitted between the rollers.

## 3 BOUNDARY CONDITIONS

With these assumptions, the normal and tangential contact problems are uncoupled and the former becomes merely an example of the classical Hertzian contact theory.

If there were no tangential elastic displacement of the surfaces, there would be a steady axial slip velocity of $\phi V$, where $V$ is the corresponding rolling velocity at the contact area. This potential slip must be taken up by elastic deformation in any region of adhesion and hence, as a point on the rollers moves through such a region, it will become progressively more displaced elastically.

Relative to a frame of reference which moves with the contact area, a point on the rollers has a velocity $-V$. It follows that the relative elastic displacement *in an adhesive region* is

$$u_x = -\phi y + f(x) \tag{1}$$

where $y$, $x$, are co-ordinates in the rolling direction and the slip direction respectively, and $f(x)$ is an arbitrary function of $x$. The origin of co-ordinates can be taken at the centre of the contact ellipse without loss of generality.

*In a region of microslip* there is no constraint on the relative tangential displacement, but the tangential traction must be $\mu p(x, y)$ and be in the same direction as the relative slip, where $\mu$ is the coefficient of friction and $p(x, y)$ is the local Hertzian contact pressure.

## 4 METHOD OF SOLUTION

The problem is very complex for general values of the misalignment angle, since (i) the relative slip (and hence the traction) in microslip regions can deviate from the axial due to elastic deformations, and (ii) the boundary between adhesive and microslip regions is difficult to determine.

However, if the misalignment angle $\phi$ is small, the contact ellipse is slender and an approximate solution can be obtained using Kalker's line integral equations (2). The method is based on an asymptotic expansion of the three-dimensional solution in terms of the slenderness ratio of the contact area. To a first approximation, the relative slip and hence the tangential traction are axial and conditions approximate to the two-dimensional solution based on local values. Since conditions vary only slowly along the axis, the first perturbation on this solution can be expressed in terms of force resultants across the contact width and only affects 'rigid body' type displacements at a given axial position.

It will be shown below that slip occurs throughout the contact area except at small values of $\phi$, for realistic values of the controlling parameters.

## 5 THE NORMAL CONTACT PROBLEM

The contact pressure, $p(x, y)$, is easily obtained from the Hertzian contact theory (see, for example, (3)) and only the results are given here. The contact ellipse has semi-axes $a$ and $b$ in the $x$ and $y$ directions respectively, where

$$a = m\left(\frac{3P(1-\nu)R_1R_2}{2G}\right)^{1/3} \tag{2}$$

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‡ *References are given in the Appendix.*

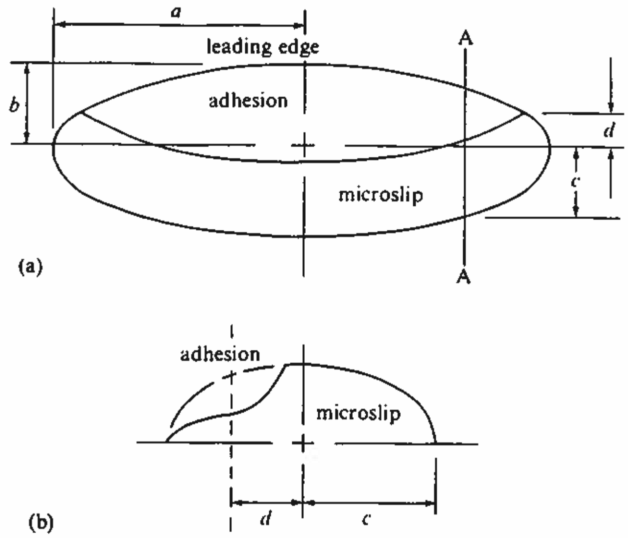


Fig. 2

than the semi-axis $b$ of the contact ellipse. It is therefore convenient to define a *slip ratio*

$$\frac{d}{b}=\frac{\pi Gab\phi}{3\mu P} \tag{15}$$

$$=\frac{\pi mn\theta}{2\mu}\left[\frac{(1-\nu)^2G(R_1+R_2)\sqrt{(R_1R_2)}}{12P}\right]^{1/3} \tag{16}$$

from equations (2), (3) and (4).

If we also define a non-dimensional load

$$P^*=\frac{2P}{(1-\nu)^2G(R_1+R_2)\sqrt{(R_1R_2)}} \tag{17}$$

and and let $P_0^*$ be the value of $P^*$ at which complete microslip occurs ($d/b=1$), we have

$$\mu^3P_0^*=(\pi mn\theta)^3/48 \tag{18}$$

from equations (16) and (17).

The parameters $m$ and $n$ are functions of $\theta$ only (see section 5 and Fig. 1 above) and, hence, equation (18) defines a relationship between $\mu^3P_0^*$ and $\theta$ which is shown in Fig. 3. An adhesive contact region is obtained if the actual normal load exceeds this critical value, in which case

$$d/b=(P_0^*/P^*)^{1/3} \tag{19}$$

It is also of interest to find the total axial force generated by the misalignment, which is

$$F=\int_{-a}^{+a}\int_{-c}^{+c}X(p,q)\,\mathrm{d}q\,\mathrm{d}p \tag{20}$$

On substituting for $X(p, q)$ and $c(p)$ from equations (10) and (11), this integral is easily evaluated to obtain the relation

$$F=\mu P\left[1-\left(1+\frac{1}{2}\frac{d^2}{b^2}\right)\left(1-\frac{d^2}{b^2}\right)^{1/2}+\frac{3d}{2b}\arccos\frac{d}{b}\right] \tag{21}$$

which is shown in Fig. 4.

It should perhaps be noted that this force acts along a line bisecting the small angle $\phi$ between the axis of the two cylinders and, hence, has a small component $1/2(F\phi)$ opposing the rolling motion of each cylinder.

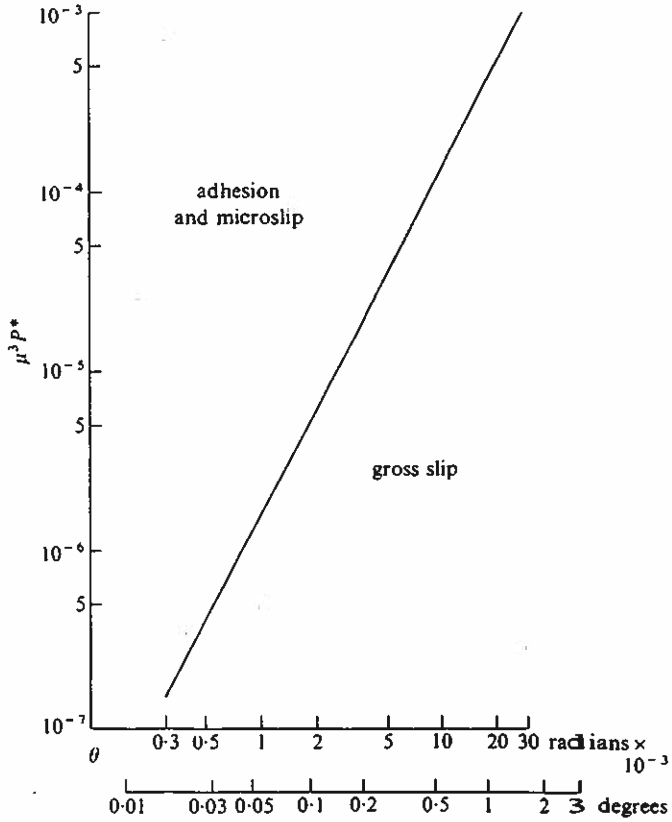


Fig. 3

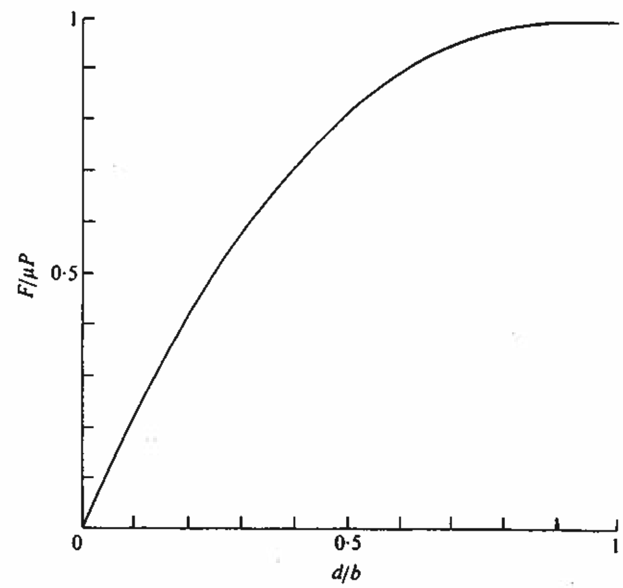


Fig. 4

Thus, a torque must be applied to each cylinder to perpetuate the motion. The power input $F\phi V$ provided by these torque is dissipated in friction in the microslip region.

## 8 CYLINDER ROLLING ON A PLANE

If one of the two solids has a plane surface ($1/R_2=0$), the contact area remains a strip for all values of $\phi$ and the above solution is not suitable. However, the

$$b = n\left(\frac{3P(1-\nu)R_1R_2}{2G}\right)^{1/3} \qquad (3)$$

$P$ is the total normal force and $\nu$ and $G$ are Poisson's ratio and the modulus of rigidity, respectively, for the material.

The non-dimensional ratios, $m$ and $n$, are functions of an auxiliary angle

$$\theta = \frac{2\phi\sqrt{(R_1R_2)}}{(R_1+R_2)} \qquad (4)$$

and can be found from the simultaneous equations

$$\frac{\pi}{2} m^3 \sin^2 \frac{\theta}{2} = \frac{K-E}{\{1-(n^2/m^2)\}} \qquad (5)$$

$$\frac{\pi}{2} n^3 \cos^2 \frac{\theta}{2} = \frac{(n/m)\{E-(n^2/m^2)K\}}{\{1-(n^2/m^2)\}} \qquad (6)$$

where $K$ and $E$ are complete elliptic integrals of complementary modulus $n/m$. The ratio between these equations gives an expression for $\tan^2 (\theta/2)$ in terms of $n/m$ and permits an inverse solution. Tables of values are given by Timoshenko and Goodier (**3**) and Kornhauser (**4**). Values of $m$ and $n$ at small values of $\theta$ required for the subsequent analysis are plotted in Fig. 1.

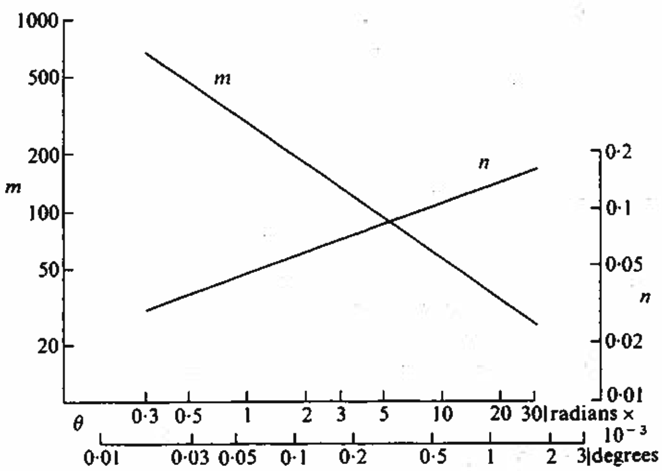


Fig. 1

The derivation of equation (4) depends upon the assumption that $\phi$ is small. For rollers of equal radius, $R$, we have $\theta = \phi$—i.e., the auxiliary angle is the same as the angle of misalignment.

The contact pressure distribution is

$$p(x, y) = \frac{3P}{2\pi ab}\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)} \qquad (7)$$

## 6 THE TANGENTIAL CONTACT PROBLEM

If the tangential traction in the $x$ direction at the point $p$, $q$ is $X$ $(p, q)$ the corresponding relative tangential surface displacement at the point $x$, $y$ will be $u_x(x, y)$ where

$$\pi G u_x(x, y) = -2\int_{-c(x)}^{+c(x)} X(x, q) \ln|q-y| \, dq + \mathrm{F}_x(x)\{\ln 4(a^2-x^2)-2\nu\} + \int_{-a}^{+a}\{F_x(p)-\mathrm{F}_x(x)\}\frac{dp}{|p-x|} + \mathrm{O}\{X \ln(a)/a^2\} \qquad (8)$$

from equation 36(b) of reference (**2**) where

$$\mathrm{F}_x(x) = \int_{-c(x)}^{+c(x)} X(x, q) \, dq \qquad (9)$$

and

$$c(x) = b\sqrt{\{1-(x^2/a^2)\}} \qquad (10)$$

defines the edge of the contact ellipse. Equation (8) differs from Kalker's by a factor of 2 since both rollers will deform giving twice the relative displacement.

We also note from Kalker's equations 36(a, b) (**2**) that the coupling between a traction in the $x$ direction and a displacement in the $y$ direction, and vice versa, is small, of the order $\{\nu Y \ln(a)/a\}$, and, hence, to a first approximation the relative slip and the traction will be purely axial.

In the adhesive region, the displacement contains the term $-\phi y$ (see equation (1)), and this can only come from the first integral on the right-hand side of equation (8), as the other terms are independent of $y$. The problem is, thus, mathematically analogous to the two-dimensional case of rolling with traction, treated by Carter (**5**) and Poritsky (**6**) and extended to arbitrary tangential loading by Heinrich and Desoyer (**7**). It can be solved by superposing a traction over the entire contact area and a traction of similar form but opposite sign to form an adhesive region adjacent to the leading edge. i.e., for $(x^2/a^2)+(y^2/b^2)<1$,

$$X(x, y) = \frac{3\mu P}{2\pi ab}\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}; \quad (y-d)^2 > (c-d)^2 \qquad (11a)$$

$$= \frac{3\mu P}{2\pi ab}\left[\sqrt{\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}} - A\sqrt{\left\{1-\frac{(y-d)^2}{(c-d)^2}\right\}}\right];$$
$$(y-d)^2 < (c-d)^2 \qquad (11b)$$

where $A$ and $d$ are functions of $x$ to be determined. Substituting equations (11) into equation (8) and integrating, we obtain

$$\pi G u_x(x, y) = \frac{3\mu P}{2a}\left[-\frac{y^2}{b^2}+\frac{A(y-d)^2}{b(c-d)}\right] + f_1(x);$$
$$(y-d)^2 < (c-d)^2 \qquad (12)$$

On comparing equations (1) and (12), it follows that the boundary conditions in the adhesive region $(y-d)^2 < (c-d)^2$ are satisfied if

$$A = (c-d)/b \qquad (13)$$

$$d = (\pi G \phi a b^2)/(3\mu P) \qquad (14)$$

The boundary conditions in the microslip region are satisfied by equation (11a) and hence equations (11), (13) and (14) define the solution to the problem.

The distance, $d$, between the centre of the adhesive region and the $x$ axis is independent of $x$ and, hence, the adhesion–microslip boundary is a reflection of the leading edge of the contact area as shown in Fig. 2(a).

Figure 2(b) shows the traction distribution defined by equations (11) at a typical cross-section.

## 7 RESULTS

Microslip must occur throughout the contact area if equation (14) defines a value of $d$ which is greater problem is easily solved in the same manner and only the principal results are given here for brevity.

The half-width of the contact strip is

$$b=\left(\frac{4P(1-\nu)R_1}{\pi G}\right)^{1/2} \tag{22}$$

where $P$ is the normal force per unit length.

The slip ratio—now simply the ratio between the widths of microslip and contact strips—becomes

$$\frac{d}{b}=\left(\frac{P_0^*}{P^*}\right)^{1/2} \tag{23}$$

where

$$P^*=\frac{P}{(1-\nu)GR_1} \tag{24}$$

and

$$\mu^2 P_0^*=\frac{\pi\phi^2}{4} \tag{25}$$

(*cf* equations (19), (17) and (18)).

The tangential traction in the axial direction in $y^2<b^2$ is

$$X(y)=\frac{2\mu P}{\pi b}\sqrt{\left(1-\frac{y^2}{b^2}\right)};\quad (y-d)^2>(b-d)^2 \tag{26a}$$

$$=\frac{2\mu P}{\pi b}\left[\sqrt{\left(1-\frac{y^2}{b^2}\right)}-\left(1-\frac{d}{b}\right)\sqrt{\left\{1-\frac{(y-d)^2}{(b-d)^2}\right\}}\right];$$
$$(y-d)^2<(b-d)^2 \tag{26b}$$

corresponding to an axial force

$$F=\mu P\left(2\frac{d}{b}-\frac{d^2}{b^2}\right) \tag{27}$$

per unit length.

The critical misalignment angle is

$$\phi_0=2\mu\sqrt{\left(\frac{P}{\pi G(1-\nu)R_1}\right)} \tag{28}$$

at and above which the axial force is

$$F=\mu P \tag{29}$$

## 9 DISCUSSION

The results of the previous two sections are presented in non-dimensional terms in the interests of generality, but it is desirable to give some indication of the angle of misalignment above which slip occurs throughout the contact area for practical systems.

From equations (19) and (23) it follows that greater angles can be tolerated at larger normal loads, but these require higher contact stresses. The critical misalignment angle for a given load can be expressed in the form

$$\phi_0=\frac{2\mu p(0)}{G} \tag{30}$$

from equations (7, 15), where $p(0)$ is the maximum contact pressure. The ratio $\{p(0)\}/G$ can be taken as a measure of the severity of the normal loading. With $\{p(0)\}/G=0{\cdot}005$ and $\mu=0{\cdot}3$, the critical angle is 0·003 radians (0·17°). Thus, extensive microslip can occur at relatively small angles of misalignment.

For rollers of similar radii, the ratio of major to minor axes of the contact ellipse is 1100 at this angle and, hence, contact may extend from end to end of a pair of rollers of finite length. Kalker's line integral method could in principle be extended to this case, but the two-dimensional solution (section 8 above) will probably give a reasonable, approximate result.

In this case, the radius $R_1$ in equations (22), (24) and (28) should be replaced by

$$R=\frac{R_1R_2}{(R_1+R_2)} \tag{31}$$

as in normal contact problems.

## 10 CONCLUSIONS

The above solution defines the distribution of tangential traction between two rolling cylinders of identical elastic materials whose axes are slightly misaligned. Slip occurs throughout the contact area for misalignment angles above a certain critical value which is related to the ratio between maximum contact pressure and elastic modulus, and is relatively small for most practical systems.

For smaller but non-zero angles of misalignment, the contact area always contains a region of microslip, but there is also a region of adhesion adjacent to the leading edge.

An axial force is generated due to misalignment and a small torque must be applied to each cylinder to perpetuate the motion.

## APPENDIX 1

### REFERENCES

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