Energy Considerations in Systems With Varying Stiffness

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If the stiffness of an elastic system changes with time, a conventional Newtonian statement of the equations of motion will generally lead to solutions that violate the fundamental mechanics principle that the work done by the external forces be equal to the increase in total energy of the system. Timoshenko's discussion of the problem of a vehicle driven across an elastic bridge is generalized to show that energy conservation can be restored only if the local deformation of the components is taken into account in determining the direction of the contact force. This result has important consequences for the interaction of elastic systems in general, including, for example, the dynamic behavior of meshing gears. [DOI: 10.1115/1.1574060]

1 Introduction

If a mechanical system contains no energy sources or dissipative mechanisms (such as friction or plasticity), the work done by the external forces must be equal to the increase in total potential energy of the system. This principle is one of the pillars of mechanics, but apparent counter examples can be produced if the system contains components whose stiffness changes with time. This inconsistency is a clear indication that the problem is in some sense ill-posed. In the present paper, we shall demonstrate that the energy conservation principle will be satisfied in such cases only if the local deformation of the components is taken into account in determining the direction of the contact force.

An important application in which the stiffness of a mechanical component varies with time concerns the meshing of two gears. In this case, the meshing stiffness changes as the contact point moves over the gear teeth and at the point where an additional pair of teeth comes into contact or leaves contact. Other examples include a vehicle moving over an elastic bridge or a loaded system in which the elastic modulus of the material changes as a function of temperature.

To introduce the subject, consider the simple case of a linear spring of stiffness k loaded by a force F. The extension of the spring, u, and the strain energy stored, U, are given by

$$u=\frac{F}{k};\quad U=\frac{F^2}{2k}, \tag{1}$$

respectively. If we now slowly change the stiffness of the spring by an amount δk (for example, by changing the temperature and hence the elastic modulus of the material), the force F will do work

$$\delta W=F\,\delta u=F\,\frac{\partial u}{\partial k}\,\delta k=-\frac{F^2\,\delta k}{k^2}, \tag{2}$$

but the strain energy will increase by

$$\delta U=\frac{\partial U}{\partial k}\,\delta k=-\frac{F^2\,\delta k}{2k^2}, \tag{3}$$

so the system appears to violate the principle of conservation of energy under a change of stiffness.

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In this example, the inconsistency will be resolved if the problem is reformulated in the context of thermodynamics and the apparent energy deficit will be associated with an exchange between thermal and mechanical energy. However, similar problems arise in purely mechanical problems, where the stiffness change is due to kinematic effects. These effects are generally not explicitly remarked in the literature. For example, the change in meshing stiffness of involute gears is sometimes approximated by representing the meshing stiffness by a sinusoidal function, [1]. The resulting equation of motion then takes the form of the Mathieu equation, which has domains of instability in which an initial perturbation from the steady periodic state will grow exponentially with time. Clearly this implies that the total energy increases with time. However, the mean power at input and output are equal and opposite, so the system as modeled violates the principle of conservation of energy.

2 A Cantilever Beam Problem

A simple example with a kinematically varying stiffness involves the cantilever beam of Fig. 1(*a*), loaded by a transverse force F at a distance x from the support. Elementary calculations show that the displacement under the force and the strain energy are

$$u=\frac{Fx^3}{3EI};\quad U=\frac{F^2x^3}{6EI}, \tag{4}$$

respectively, where EI is the flexural rigidity of the beam. If the point of application of the force now moves from x to $x+\delta x$, the displacement will change by

$$\delta u=\frac{\partial u}{\partial x}\,\delta x=\frac{Fx^2\,\delta x}{EI}, \tag{5}$$

allowing the forces to do work

$$\delta W=\frac{F^2x^2\,\delta x}{EI}. \tag{6}$$

However, the corresponding change in strain energy in the beam is only

$$\delta U=\frac{\partial U}{\partial x}\,\delta x=\frac{F^2x^2\,\delta x}{2EI}. \tag{7}$$

Alternatively, these results can be obtained from Eqs. (1)–(3) by substituting $k=3EI/x^3$.

This paradox is related to that remarked by Timoshenko and others, [2–6], in connection with the vibration of beams subject to moving transverse loads. If a vehicle drives across a bridge supported at both ends, the gravitational force does no net work, since the vehicle leaves at the same vertical level as it enters, but in

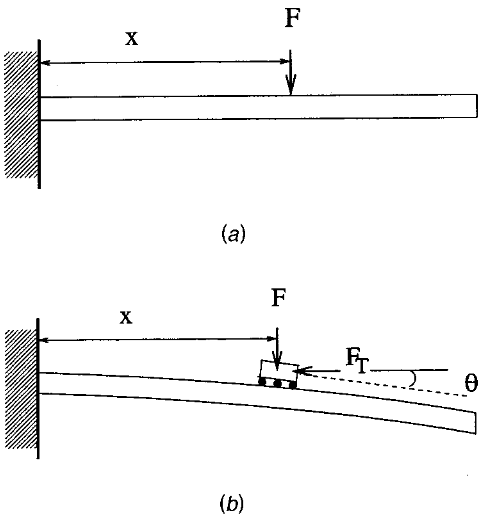

Fig. 1 (*a*) The cantilever beam loaded by a normal force, (*b*) the same beam loaded through a frictionless roller

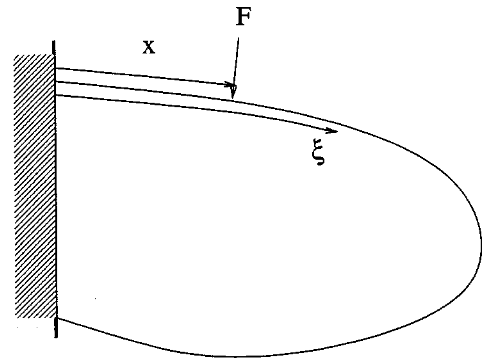

Fig. 2 General elastic structure loaded by a normal contact force

general the bridge will be left in a state of vibration and hence will be in a higher energy state than it was before the transit. Where does the extra energy come from?

Timoshenko resolved the problem by noting that the instantaneous motion of the vehicle is not horizontal because of the deflected shape of the bridge. It follows that the brakes or the engine must be engaged to ensure a constant transit speed and this introduces additional energy terms. This argument can be applied to our cantilever beam problem by introducing the modified system of Fig. 1(*b*), in which the force is transmitted to the beam through a roller. If the roller is frictionless, it can be retained in equilibrium only by the application of a tangential force

$$F_T = F \tan\theta, \tag{8}$$

where

$$\tan\theta = \frac{Fx^2}{2EI} \tag{9}$$

is the slope of the beam at the point of application of the force. If the roller in Fig. 1(*b*) moves a distance δx to the right, an amount of work

$$\delta W_T = F_T \delta x = \frac{F^2 x^2 \delta x}{2EI} \tag{10}$$

will be done against the force F_T and the inclusion of this term completes the energy balance

$$\delta U = \delta W - \delta W_T. \tag{11}$$

Lee [3] showed that the same conclusion could be achieved without recourse to arguments from contact mechanics. We adapt the notion of Lee (who uses a convected time derivative) to the quasi-static case under consideration here. We decompose the motion of the force into two processes. In the first phase, the beam is "frozen" in its deformed state while the force moves from x to $x+\delta x$. During this phase, the force moves a distance $\delta u_1 = \delta x \tan\theta$ and hence does work,

$$\delta W_1 = F\,\delta u_1 = F\,\delta x \tan\theta = \frac{F^2 x^2 \delta x}{2EI}, \tag{12}$$

but none of this work is communicated to the beam. In the second phase, the beam is allowed to relax to its new equilibrium position. The additional displacement of the force is $\delta u_2 = \delta u - \delta u_1$ and the work done during this phase

$$\delta W_2 = F\,\delta u_2 = \frac{F^2 x^2 \delta x}{2EI} \tag{13}$$

is communicated to the beam and results in the increase of strain energy δU.

Notice that if the direction of motion is reversed, the quantity δW_1 will be negative, showing that an external source of energy is required in phase 1 to move the force over the "frozen" beam.

Of course, in a real physical application, the work δW_1 cannot simply be lost to or generated from a fictitious energy source and any practical realization of the problem will bring us back into the realm of contact mechanics.

3 A More General Case

Figure 2 shows a more general elastic structure loaded by a normal force F at a point on the boundary characterized by a curvilinear coordinate x. We assume that strains and rotations are small, but the elastic behavior is not necessarily linear. We also assume that the force produces a bounded displacement at its point of application. This restriction will be removed for the linear case in Section 4.

We define the normal displacement at a general point on the boundary $x=\xi$ as

$$u = u(F,x,\xi), \tag{14}$$

in which case the local rotation of the deformed surface is

$$\theta(\xi) = \frac{\partial u}{\partial \xi}(F,x,\xi). \tag{15}$$

We also define the functions

$$f(F,x) \equiv u(F,x,x); \quad g(F,x) = \theta(F,x,x), \tag{16}$$

which are the normal displacement and rotation at the point of application of the force.

The strain energy in the structure can be found by applying F gradually, keeping its location fixed. It is therefore given by

$$U(F,x) = \int_0^F \frac{\partial u(F,x,x)}{\partial F} F\,dF = \int_0^F \left(\frac{\partial f}{\partial F}\right) F\,dF. \tag{17}$$

If, following Timoshenko's scenario, the force is applied through a frictionless roller, we shall require a restraining force

$$F_R = F\theta(F,x,x) = Fg(F,x). \tag{18}$$

If the roller is now allowed to move a distance δx, the force F will do an increment of work

$$\delta W = F \frac{\partial f}{\partial x} \delta x, \tag{19}$$

but F_R will have work done against it equal to

$$\delta W_R = F_R \delta x = Fg(F,x)\delta x. \quad (20)$$

Thus, the net work done on the structure will be

$$\delta W - \delta W_R = \left(F \frac{\partial f}{\partial x} - Fg \right) \delta x. \quad (21)$$

Equating this to the increase in strain energy in the structure, we obtain

$$\frac{\partial U}{\partial x} = F \frac{\partial f}{\partial x} - Fg = \int_0^F \left(\frac{\partial^2 f}{\partial F \partial x} \right) F dF, \quad (22)$$

from Eq. (4). Differentiating with respect to F, we then obtain

$$F \frac{\partial^2 f}{\partial F \partial x} = F \frac{\partial^2 f}{\partial F \partial x} + \frac{\partial f}{\partial x} - g - F \frac{\partial g}{\partial F} \quad (23)$$

and hence

$$\frac{\partial f}{\partial x} - g - F \frac{\partial g}{\partial F} = 0, \quad (24)$$

which defines a relationship between the displacement and the slope of the structure which must be satisfied if energy is to be conserved during the process.

Alternative Proof. An alternative proof of this result can be obtained by invoking the incremental form of Maxwell's reciprocal theorem, [7], for small perturbations about the reference state where the force F is applied at x, producing displacement u and rotation θ.

Moving F by a distance δx is equivalent to adding an infinitesimal moment $\delta M = F \delta x$ at x. The response of the structure can be linearized for small perturbations about the reference state, leading to the relation

$$\frac{\partial u}{\partial M} = \frac{\partial \theta}{\partial F} = \frac{\partial g}{\partial F}. \quad (25)$$

Following Lee's argument above, we consider the displacement due to the motion of the force from x to $x + \delta x$ as the sum of two parts. Freezing the beam in its deformed shape, we have $\delta u_1 = \theta(F,x,x)\delta x$ after which relaxation to the new equilibrium position gives an additional displacement δu_2 associated with the moment $F\delta x$. The total displacement of the force along its line of action is therefore

$$\theta(F,x,x)\delta x + \frac{\partial u}{\partial M} F \delta x = g(F,x)\delta x + \frac{\partial g}{\partial F} F \delta x, \quad (26)$$

but this displacement is also given by

$$\frac{\partial f}{\partial x} \delta x,$$

giving

$$\frac{\partial f}{\partial x} = g + F \frac{\partial g}{\partial F} \quad (27)$$

as before.

4 Distributed Forces

In problems of linear elasticity, the displacement field due to a concentrated force is singular at the point of application of the force and hence the functions f, g of the previous section are not well defined. However, the concentrated force solution can still be used as a Green's function to define the effect of a distributed force by superposition. Consider the case where Fig. 2 represents a two-dimensional linear elastic body and suppose that a concentrated normal force F at x produces a normal displacement u at ξ, where

$$u(\xi) = F\tilde{u}(x,\xi). \quad (28)$$

Now consider the case where the force is distributed in the vicinity of x, with intensity

$$p(x+r) = Ff(r), \quad (29)$$

where the distribution function $f(r)$ is nonzero only in $-c < r < c$ and is normalized so that

$$\int_{-c}^{c} f(r) dr = 1. \quad (30)$$

The displacement due to this distribution can now be written by superposition as

$$u(\xi) = F \int_{-c}^{c} \tilde{u}(x+r,\xi) f(r) dr \quad (31)$$

and the local slope is

$$\theta(\xi) = \frac{\partial u}{\partial \xi} = F \int_{-c}^{c} \frac{\partial \tilde{u}}{\partial \xi}(x+r,\xi) f(r) dr. \quad (32)$$

The stored strain energy is

$$U = \frac{1}{2} \int_{x-c}^{x+c} u(\xi) p(\xi) d\xi$$
$$= \frac{F^2}{2} \int_{-c}^{c} \int_{-c}^{c} \tilde{u}(x+r,x+s) f(r) f(s) dr ds, \quad (33)$$

writing $\xi = x+s$. If the distributed force is now displaced a distance δx, the increment in strain energy will be

$$\delta U = \frac{\partial U}{\partial x} \delta x$$
$$= \frac{F^2 \delta x}{2} \int_{-c}^{c} \int_{-c}^{c} \left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{u}}{\partial \xi} \right) (x+r,x+s) f(r) f(s) dr ds. \quad (34)$$

The additional work done by the normally directed distributed force during this motion is

$$\delta W = \delta x \int_{x-c}^{x+c} \left(\theta(\xi) + \frac{\partial u}{\partial x}(\xi) \right) p(\xi) d\xi$$
$$= F^2 \delta x \int_{-c}^{c} \int_{-c}^{c} \left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{u}}{\partial \xi} \right) (x+r,x+s) f(r) f(s) dr ds, \quad (35)$$

which is exactly twice the increment of strain energy. The remaining work is required to overcome the implied tangential restraining force F_T, which is the resultant of tractions equal and opposite to the component of p parallel to the *deformed* surface and is given by

$$F_T = \int_{x-c}^{x+c} \theta(\xi) p(\xi) d\xi$$
$$= F^2 \int_{-c}^{c} \int_{-c}^{c} \frac{\partial \tilde{u}}{\partial \xi}(x+r,x+s) f(r) f(s) dr ds. \quad (36)$$

Notice that from Maxwell's reciprocal theorem we have

$$\tilde{u}(x,\xi) = \tilde{u}(\xi,x); \quad \frac{\partial \tilde{u}}{\partial \xi}(x+r,x+s) = \frac{\partial \tilde{u}}{\partial x}(x+s,x+r) \quad (37)$$

and hence

$$\int_{-c}^{c} \int_{-c}^{c} \frac{\partial \tilde{u}}{\partial \xi}(x+r,x+s) f(r) f(s) dr ds$$
$$= \int_{-c}^{c} \int_{-c}^{c} \frac{\partial \tilde{u}}{\partial x}(x+r,x+s) f(r) f(s) dr ds, \quad (38)$$

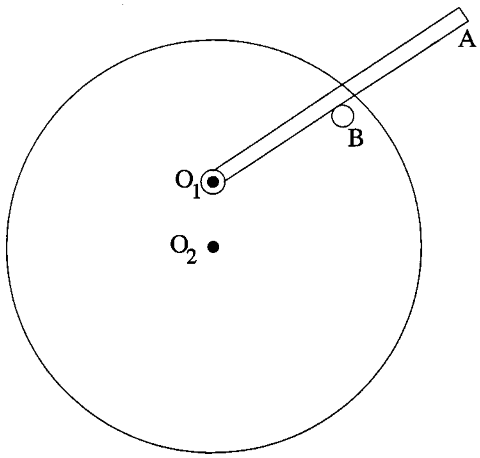

Fig. 3 Simple gear system using the cantilever beam of Fig. 1

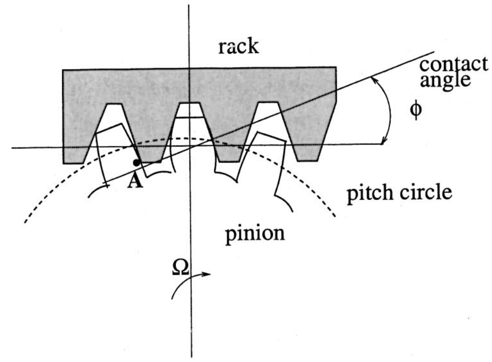

Fig. 4 A flexible involute gear meshing with a rigid rack

on interchanging the dummy variables *s*, *r*. Using this result in Eqs. (34), (35), we see that the system satisfies the principle of conservation of energy[1] if and only if we include the contribution of the tangential force F_T.

5 Implications in Contact Mechanics

These results show that in systems with kinematically varying stiffness, the direction of the contact forces or tractions must be chosen to be normal to the *deformed* contact surface in a Newtonian statement of the problem if energy conservation is to be preserved. This contradicts the conventional wisdom in contact mechanics, where the direction of the contact forces is referred to the undeformed configuration of the contacting bodies.

For dynamic systems involving moving contacts such as the meshing of two gears, failure to include this effect will generally lead to equations of motion that are incorrect because their solution violates the fundamental principle that the work done by the external forces equal the change in total potential energy of the system.

Figure 3 shows a simple illustrative example in which a "gear" comprising a flexible beam O_1A rotates clockwise at constant speed Ω about a center O_1 that is fixed in space. The rod drives a rigid pin B mounted on a rigid disk which rotates about center O_2. If O_1, O_2 are not coincident, the effective length O_1B of the beam will vary with angular position, leading to a kinematically varying stiffness.

The beam support can be brought to rest by superposing a counterclockwise rigid-body rotation Ω on the whole system. The disk will then be seen to execute a more complex motion whose effect however is merely to cause the pin B to slide quasi-sinusoidally along the beam. The results of Section 2 therefore show that the correct (energy conserving) equations of motion for this system will be obtained only if the local slope of the deflected beam is taken into account in determining the direction of the transmitted force and hence the torque transmitted by the disk O_2B.

5.1 Involute Gears. Similar considerations apply to the more complex system of the meshing of two involute gears. Once again, there will generally be a variation of effective stiffness as the contact point moves along the surfaces of the two meshing teeth. However, most gear systems will have a noninteger contact ratio, implying that the number of teeth in contact changes during the meshing cycle, resulting in a major change in contact stiffness. The present energy arguments show that there must be a sudden input of energy to the system at the point where the number of teeth in contact (and hence the total contact stiffness) *increases* and a sudden removal of energy when it *decreases*. The only external forces acting on the system that do work are the input and output torques, so we must conclude that there will be a significant change in the instantaneous torque ratio of the gears during these transitions, associated with the change in the line of action of the contact forces due to gear tooth deformation. The reader is invited to conduct a 'thought experiment' in which the output (reaction) torque is held constant and the input shaft is rotated at extremely slow speed. A significant spike will be needed in the input torque to force an extra tooth into contact. By contrast, when a tooth leaves contact, the input shaft will tend to spring ahead.

To explain this result, notice first that the tooth deflection causes a relative rigid-body motion of the gears, so at the point where an additional tooth would theoretically come into contact as a result of involute action, the new unloaded tooth would actually be in a position implying interpenetration. This is illustrated in Fig. 4 for a deformable gear meshing with a rigid rack. As a result, the contact of the new tooth will actually start before the theoretically correct point and it will involve contact of the noninvolute corner of the rack (Point A in Fig. 4) with the flank of the tooth. The line of action of the transmitted force will deviate considerably from the theoretical pressure line during this engagement period as a consequence of local tooth deformation. The analysis of this problem would be geometrically complex, but the results of Sections 3 and 4 show that the use of the true direction of the contact force would lead to the same result as the simpler energy-based analysis.

The tooth engagement period represents only a small proportion of the tooth period and an acceptable idealization in many cases is to assume it is instantaneous, leading to discontinuities (jumps) in stiffness. However, this implies the occurrence of discontinuities in strain energy and energy conservation requires corresponding discontinuities in kinetic energy and hence in rotational speed. The implications of these discontinuities for the dynamics of involute gear sets will be discussed in a separate paper, [8].

6 Conclusions

In this paper we have shown that a conventional statement of the equations of motion for a system with kinematically varying stiffness will generally lead to a solution that violates the fundamental mechanics principle that the work done by the external forces be equal to the change in total potential energy of the system. We cannot emphasize too strongly that such equations of motion are therefore incorrect.

To obtain a correct statement of the governing differential equations, it is necessary to allow for the local deformation of the components in determining the direction of the contact forces or tractions. This result, which is a generalization of the "Timosh-

[1]Notice that for an elastic body, the tangential force F_T will also induce a local tangential deformation, but the associated work term is of second order relative to those considered above.

enko paradox,'' applies even in small strain problems where the problem statement is conventionally referred to the undeformed configuration.

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