Finite Element Analysis of Thermoelastic Contact Stability

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When heat is conducted across an interface between two dissimilar materials, thermoelastic distortion affects the contact pressure distribution. The existence of a pressure-sensitive thermal contact resistance at the interface can cause such systems to be unstable in the steady-state. Stability analysis for thermoelastic contact has been conducted by linear perturbation methods for one-dimensional and simple two-dimensional geometries, but analytical solutions become very complicated for finite geometries. A method is therefore proposed in which the finite element method is used to reduce the stability problem to an eigenvalue problem. The linearity of the underlying perturbation problem enables us to conclude that solutions can be obtained in separated-variable form with exponential variation in time. This factor can therefore be removed from the governing equations and the finite element method is used to obtain a time-independent set of homogeneous equations in which the exponential growth rate appears as a linear parameter. We therefore obtain a linear eigenvalue problem and stability of the system requires that all the resulting eigenvalues should have negative real part. The method is discussed in application to the simple one-dimensional system of two contacting rods. The results show good agreement with previous analytical investigations and give additional information about the migration of eigenvalues in the complex plane as the steady-state heat flux is varied.

1 Introduction

If heat is conducted across an interface between two elastic bodies, thermoelastic distortion generally affects the contact pressure and the extent of the contact area (Clausing, 1966; Jones et al., 1975). Early solutions of such problems encountered difficulties with existence of solution (Lewis and Perkins, 1968; Barber, 1973), which were later resolved by the use of a more physically realistic boundary condition in which a pressure-dependent thermal resistance exists at the interface (Barber, 1978).

Even with this refinement, the steady-state solution can be nonunique and also in some circumstances unstable (Duvaut, 1979; Shi and Shillor, 1990). Experimental evidence of instability of practical thermoelastic contact systems has been reported for heat exchanger tubes (Srinivasan and France, 1985) and for the solidification of castings (Richmond and Huang, 1977).

Analysis of the stability of thermoelastic contact was first attempted for one-dimensional systems, using linear perturbation methods (Barber et al., 1980; Barber 1981, 1986). The method was later extended to the two-dimensional problem of the contact of two-half planes (Barber, 1987, Zhang and Barber, 1990), making use of the fact that an arbitrary perturbation could be represented as a superposition of spatially sinusoidal terms, each of which could be represented in separated-variable form, thus reducing the problem essentially to one dimension. The same method has been used to consider the problem of a layer of finite thickness in contact with a half-plane (Yeo and Barber, 1991). Difficulties of a different order are encountered when the contact area is bounded, since separated-variable solutions cannot then generally be used. In such cases, methods using series solutions might be envisaged, but it seems likely that a direct numerical treatment would be more efficient. In the present paper, we therefore develop one possible numerical implementation for the perturbation problem and illustrate its use in the simple two-rod geometry of Barber and Zhang (1988).

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2 The Finite Element Method

An obvious numerical approach to the problem would be to simulate the transient behavior of the system—for example, using a finite element description of the temperature and stress fields in space and time. Parametric investigation of such a system would then enable stability boundaries to be determined, as well as identifying stable and unstable steady-states, in cases of multiple solution. However, such an effort would be extremely computer-intensive, since (i) the

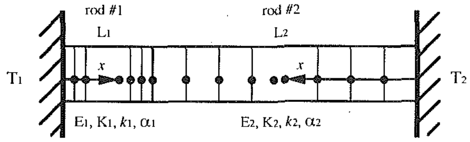

Fig. 1 The two-rod problem

solution of the transient heat conduction equation with reasonably fine meshing involves the use of a very small time increment for numerical stability and convergence and (ii) the additional effect of a very pressure-sensitive boundary condition at the contact interface would create considerable computational difficulties of its own. Similar difficulties are encountered with other numerical implementations of transient thermoelastic contact problems and are documented in (Barber and Zhang, 1988; Azarkhin and Barber, 1986).

A preferable alternative is suggested by the fact that the system (including the boundary conditions) is linear to sufficiently small perturbations from the steady-state. It therefore follows that if we postulate the existence of a separated-variable solution for the perturbation in which the fields grow with time t according to $\exp(bt)$, the exponential term will cancel in the governing equations, leaving a time-independent problem that can be solved by the finite element method. The resulting equations will be homogeneous and will have a nontrivial solution only for certain eigenvalues of b. We can then construct a more general solution to the transient problem as a series of the associated eigenfunctions. Assuming this eigenfunction series to be complete, we can deduce that the system will be stable if and only if all the eigenvalues have negative real part.

2.1 The Two-Rod Problem. The ideas behind the method are best explained in terms of its implementation for the simple two-rod model, which was treated by analytical methods by Barber and Zhang (1988). The results of this latter reference also give a convenient check on the validity of the proposed method.

The system is illustrated in Fig. 1. Two perfectly conducting rigid walls are maintained at constant temperatures T_1, T_2, respectively. Two uniform elastic rods of lengths L_1, L_2 are built into the respective walls and make contact at their free ends. Young's modulus, thermal conductivity, thermal diffusivity and coefficient of thermal expansion for the materials are denoted by E_γ, K_γ, k_γ, α_γ respectively ($\gamma = 1,2$).

Heat flow occurs across the contacting interface at the free ends of the rods through a thermal contact resistance (Shlykov and Ganin, 1964; Cooper et al., 1969; Thomas and Probert, 1970) R, which is assumed to be a continuous function of the contact pressure p. This boundary condition is appropriate to situations in which the steady-state under consideration involves contact between the rods. Steady-states involving separation between the ends of the rods can be subsumed under the same analysis by defining an extension of the concept of contact pressure as in Eq. (7) of Barber and Zhang (1988).

We investigate the circumstances in which a small perturbation from the steady-state can grow with time. Because the perturbation is small, the relation

$$T = QR \tag{1}$$

between the heat flux, Q, at the interface, the temperature difference, T between the free ends of the rods and the contact resistance, R can be linearized, giving

$$\Delta T = Q_0 \Delta R + R_0 \Delta Q, \tag{2}$$

where Q_0, T_0, R_0 denote the corresponding steady-state values and ΔQ, ΔT, ΔR are the perturbations. Noting that $\Delta R = R'(\mathrm{p}_0)\Delta \mathrm{p}$, we can rewrite (2) as

$$\Delta T = Q_0 R'(p_0)\Delta p + R_0 \Delta Q. \tag{3}$$

Equation (3) is a linear boundary condition and hence the perturbation is governed entirely by linear equations, the nature of which permit a separated-variable solution in which the time variation is of the form $\exp(bt)$ where b is a (possibly complex) constant. We therefore postulate the existence of such a solution and examine the conditions under which it is possible. By analogy with related problems, we anticipate that the problem will reduce to a system of homogeneous linear algebraic equations with a denumerable set of eigenvalues for b. We also anticipate that the corresponding set of eigenfunctions will be complete in the sense that an arbitrary initial condition could be expressed as an eigenfunction expansion. This being the case, the stability of the system will be determined by the condition that all the eigenvalues should have negative real part.

2.2 The Heat Conduction Equation. The governing equations are only time-dependent through the heat conduction equation, which in one dimension takes the form

$$T_{,xx} = \frac{1}{k} T_{,t}. \tag{4}$$

Thus, if we assume the perturbed temperature field has the form

$$T(x,t) = \Theta(x) e^{bt}, \tag{5}$$

the function Θ of the spatial coordinates only must satisfy the equation

$$\Theta_{,xx} - \frac{b}{k}\Theta = 0. \tag{6}$$

To develop a finite element formulation of this equation, we use Galerkin's method (Hughes (1987)). Briefly, we multiply Eq. (6) by an arbitrary function $w(x)$ and integrate over the domain of each rod, obtaining for example

$$\int_0^{L_\gamma} \left(\Theta^\gamma_{,xx} - \frac{b}{k_\gamma}\Theta^\gamma \right) w^\gamma(x) dx = 0. \tag{7}$$

This must be satisfied for all functions $w^\gamma(x)$ and a system of equations can be obtained by defining the functions Θ^γ, w^γ in the discrete form

$$\Theta^\gamma(x) = \sum_{i=1}^{n} \Theta_i^\gamma N_i(x); \quad w^\gamma(x) = \sum_{i=1}^{n} w_i^\gamma N_i(x) \tag{8}$$

where $N_i(x)$ is a suitable shape function.

Integrating Eq. (7) by parts, noting that the perturbation Θ^γ must be zero at the wall ($x = 0$) and substituting for Θ^γ, w^γ from (8), we obtain

$$\mathbf{K}^\gamma \boldsymbol{\Theta}^\gamma + b\mathbf{H}^\gamma \boldsymbol{\Theta}^\gamma = \boldsymbol{\Phi}^\gamma \{q_0^\gamma\}; \quad \gamma = 1,2 \tag{9}$$

where the matrices

$$K_{ij}^\gamma = \int_0^{L_\gamma} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx;$$

$$H_{ij}^\gamma = \int_0^{L_\gamma} \frac{N_i N_j dx}{k_\gamma}; \quad \Phi_i^\gamma = -\frac{N_n(L_\gamma)\,\delta_{in}}{K_\gamma}, \tag{10}$$

δ_{in} is the Kronecker delta and $q_0^\gamma \exp(bt)$ is the perturbation in the outward heat flux at the free end of rod γ. Continuity of heat flux at the interface then demands that $\{q_0^1\} = -\{q_0^2\} \equiv \{q_0\}$.

2.3 The Thermoelastic Contact Problem. An essentially similar formulation can be used for the thermoelastic deformation and contact of the rods. Equilibrium considerations demand that the axial stress, σ_{xx}^γ be independent of x—i.e., $\sigma_{xx,x}^\gamma = 0$ and, using the thermoelastic constitutive law, we can write this condition in the form

$$E_\gamma(u^\gamma_{,xx} - \alpha^\gamma \Theta^\gamma_{,x}) = 0, \tag{11}$$

where we have already removed a factor $\exp(bt)$, so that u^γ is here to be interpreted such that the axial displacement of the rod is $u^\gamma(x)\exp(bt)$.

The resulting set of equations can be written in the form

$$\mathbf{L}^\gamma \mathbf{u}^\gamma = \mathbf{F}^\gamma \boldsymbol{\Theta}^\gamma + \mathbf{f}^\gamma\{\sigma_0\} \tag{12}$$

where

$$L_{ij}^\gamma = \int_0^{L_\gamma} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} dx;$$

$$F_{ij}^\gamma = \alpha_\gamma \int_0^{L_\gamma} \frac{\partial N_i}{\partial x} N_j dx; \quad f_i^\gamma = \frac{N_n(L_\gamma)\,\delta_{in}}{E_\gamma} \tag{13}$$

and $\sigma_0^\gamma \exp(bt)$ is the perturbation in the traction at the free end of rod γ.

It is convenient to partition this set of equations as

$$\begin{bmatrix} \mathbf{L}_{11}^\gamma & \mathbf{L}_{12}^\gamma \\ \mathbf{L}_{21}^\gamma & \mathbf{L}_{22}^\gamma \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1^\gamma \\ \mathbf{u}_2^\gamma \end{Bmatrix} = \begin{bmatrix} \mathbf{F}_1^\gamma \\ \mathbf{F}_2^\gamma \end{bmatrix} \{\boldsymbol{\Theta}^\gamma\} + \begin{Bmatrix} \mathbf{f}_1^\gamma \\ \mathbf{f}_2^\gamma \end{Bmatrix} \{\sigma_0\} \tag{14}$$

where $\mathbf{u}_2^\gamma$ is a vector with only one component representing the nodal displacement at the free (contacting) end and $\mathbf{u}_1^\gamma$ is a vector with $(n-1)$ components representing the displacements at the interior nodes.

Since the two rods are in contact at their free ends, we have

$$\mathbf{u}_2^1 = -\mathbf{u}_2^2 = \{u_0\}; \quad \{\sigma_0^1\} = \{\sigma_0^2\} = \{\sigma_0\}. \tag{15}$$

Eliminating $\mathbf{u}_1^\gamma$ from the partitioned Eqs. (14) and using (15), we obtain

$$\left(\mathbf{L}_{21}^\gamma (\mathbf{L}_{11}^\gamma)^{-1} \mathbf{F}_1^\gamma - \mathbf{F}_2^\gamma\right)\boldsymbol{\Theta}^\gamma + (-1)^\gamma \left(\mathbf{L}_{21}^\gamma (\mathbf{L}_{11}^\gamma)^{-1} \mathbf{L}_{12}^\gamma - \mathbf{L}_{22}^\gamma\right)\{u_0\} = \mathbf{f}_2^\gamma\{\sigma_0\}. \tag{16}$$

We can then eliminate u_0 between the two equations (16 with $\gamma = 1,2$) obtaining

$$\{\sigma_0\} = \mathbf{S}_1 \boldsymbol{\Theta}^1 + \mathbf{S}_2 \boldsymbol{\Theta}^2, \tag{17}$$

where

$$\mathbf{S}_1 = \mathbf{r}\left(\mathbf{r}\mathbf{f}_2^1 + \mathbf{f}_2^2\right)^{-1}\left(\mathbf{L}_{21}^1 (\mathbf{L}_{11}^1)^{-1} \mathbf{F}_1^1 - \mathbf{F}_2^1\right);$$

$$\mathbf{S}_2 = \left(\mathbf{r}\mathbf{f}_2^1 + \mathbf{f}_2^2\right)^{-1}\left(\mathbf{L}_{21}^2 (\mathbf{L}_{11}^2)^{-1} \mathbf{F}_1^2 - \mathbf{F}_2^2\right) \tag{18}$$

and

$$\mathbf{r} = \left(\mathbf{L}_{21}^2 (\mathbf{L}_{11}^2)^{-1} \mathbf{L}_{12}^2 - \mathbf{L}_{22}^2\right)\left(\mathbf{L}_{21}^1 (\mathbf{L}_{11}^1)^{-1} \mathbf{L}_{12}^1 - \mathbf{L}_{22}^1\right)^{-1}. \tag{19}$$

2.4 Stability Analysis. To complete the formulation of the stability problem, we must couple the solutions of the heat conduction and thermoelastic problems through the linearized boundary condition (3), which can be written in the matrix form

$$\mathbf{P}\boldsymbol{\Theta}^1 - \mathbf{P}\boldsymbol{\Theta}^2 = -Q_0 R'\{\sigma_0\} + R_0\{q_0\}, \tag{20}$$

where $\mathbf{P}$ is a $1 \times n$ matrix whose elements are zero except for the nth element which is unity.

Eliminating q_0, σ_0 between Eqs. (9), (17), (20) we finally obtain

$$\begin{bmatrix} R_0\mathbf{K}^1 - Q_0 R' \boldsymbol{\Phi}^1 \mathbf{S}_1 - \boldsymbol{\phi}^1 \mathbf{P} & -Q_0 R' \boldsymbol{\Phi}^1 \mathbf{S}_2 + \boldsymbol{\Phi}^1 \mathbf{P} \\ Q_0 R' \boldsymbol{\Phi}^2 \mathbf{S}_1 + \boldsymbol{\Phi}^2 \mathbf{P} & R_0 \mathbf{K}^2 + Q_0 R' \boldsymbol{\Phi}^2 \mathbf{S}_2 - \boldsymbol{\Phi}^2 \mathbf{P} \end{bmatrix} \times \begin{Bmatrix} \boldsymbol{\Theta}^1 \\ \boldsymbol{\Theta}^2 \end{Bmatrix} = b \begin{bmatrix} -R_0 \mathbf{H}^1 & 0 \\ 0 & -R_0 \mathbf{H}^2 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Theta}^1 \\ \boldsymbol{\Theta}^2 \end{Bmatrix}. \tag{21}$$

This matrix equation constitutes a generalized linear eigenvalue problem for the exponential growth rate b, which is easily solved by standard numerical methods. Instability is indicated by the system having at least one eigenvalue with positive real part.

Table 1

Rod No.	Material	E	K	k	α
		GPa	$W/m°C$	m^2/s	$°C^{-1}$
1	Stainless steel	190	21	5.93×10^{-6}	14×10^{-6}
2	Aluminum alloy	72	173	67×10^{-6}	22×10^{-6}

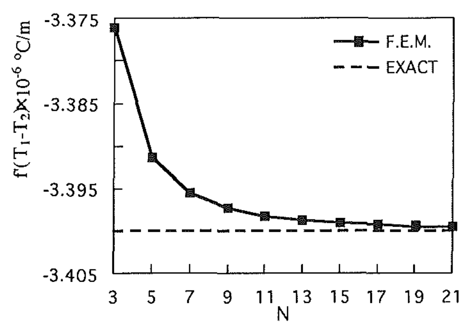

Fig. 2 Convergence of the dimensionless heat flux at the stability boundary for $L_1 = 0.1$ m, $L_2 = 0.25$ m and the steady-state contact resistance $R_0 = L_1/K_1 + L_2/K_2$

3 Results

The principle purpose of the present investigation is to test the viability of the proposed algorithm against previously published analytical results (Barber and Zhang (1988)). The rod lengths and material properties were therefore chosen to be the same as in the earlier paper, the relevant material properties being listed in Table 1. Figure 2 shows the dimensionless heat flux at the stability boundary for $L_1 = 0.1$ m, $L_2 = 0.25$ m, and a steady-state contact resistance equal to the sum of the combined resistances of the two rods—i.e., $R_0 = L_1/K_1 + L_2/K_2$, which corresponds to $f = 0.5$ in the notation of Barber and Zhang (1988). Results are shown for various values of N and compared with the exact value from the earlier analysis. Even with $N = 3$—i.e., using only three linear elements in the discretization of each rod—the stability boundary is predicted with an accuracy of better than 0.5 percent and a modest refinement of the mesh permits this estimate to be improved by an order of magnitude.

Numerical results were obtained over the entire range of parameter values treated by the analytical perturbation method by Barber and Zhang (1988) and in all cases the stability boundary, defined by the minimum steady-state heat flux required for the system to be unstable, agreed within ± 1 percent for $N = 5$.

One advantage of the present method is that it generates values for all the eigenvalues of the discretized problem and therefore permits us to investigate the migration of eigenvalues in the complex plane as the imposed steady-state heat flux is varied. Of course, the discrete problem only has $2N$ eigenvalues, as compared with a denumerably infinite set for the real problem, but the discrete eigenvalues should give good approximations for the lower terms in the set and order of magnitude estimates for those near the truncation limit.[1]

Figure 3 shows the locus of the first eigenvalue for a two-rod system in which stability is determined by migration of an eigenvalue across the imaginary axis (giving oscillatory transient behavior). Notice that the two leading eigenvalues

[1]The agreement between the stability boundary as determined by analytical and numerical methods is a measure of the high degree of accuracy of the lowest eigenvalue in the discrete problem, since this is the term that determines stability of the system.

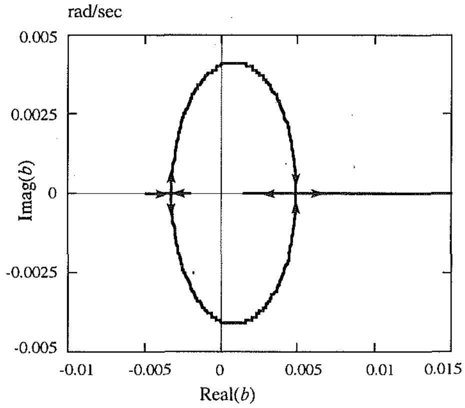

Fig. 3 Migration of the leading pair of eigenvalues, b as the heat flux Q_0 is increased for the case where $L_1 = 0.1$ m, $L_2 = 0.2$ m and the steady-state contact resistance $R_0 = L_1/K_1 + L_2/K_2$

are real and negative at low values of Q_0, but as Q_0 increases, they pass into complex values, cross the imaginary axis into the unstable half-plane and later rejoin the real axis. For the same combinations of rod lengths but the opposite direction of heat flow, the leading eigenvalues remained real for all heat fluxes and the stability boundary was therefore determined by the passage of the first eigenvalue through the origin.

It was found in all cases that at most one pair of eigenvalues passed into the right half-plane as Q_0 increased. All the remaining eigenvalues were located at substantially larger distances from the origin in the left half-plane. Thus, instability is in all cases associated with a single form of perturbation and has (possibly complex) pure exponential growth as long as the perturbation is small enough for the linearity condition to hold.

4 Conclusions

We have shown that by assuming an exponential variation in time for a linear perturbation from the steady-state solution, a finite element discretization can be developed for the spatial variation of the temperature and stress fields, leading eventually to a linear eigenvalue problem for the exponential growth rate. The results agree closely with those obtained by analytical methods for the two-rod problem. Furthermore, since stability is determined by the leading eigenvalue of the system, good accuracy can be obtained using a very modest number of elements. The proposed algorithm therefore presents an attractive approach to finite geometry thermoelastic stability problems in two and three dimensions, where analytical methods would be prohibitively complicated. These applications will be explored in a subsequent publication.

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