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Effect of Material Properties on the Stability of Static Thermoelastic Contact

When heat is conducted across an interface between two different materials, the interaction between thermoelastic distortion and thermal contact resistance can cause the system to be unstable. This paper investigates the influence of material properties on the stability criterion for an interface between two half planes. It is found that most material combinations exhibit one or other of two kinds of stability behavior. In one of these, the stability criterion is closely related to that for uniqueness of the steady-state solution and instability is only possible for one direction of heat flow. In the other, instability can occur for either direction of heat flow and in one case is characterized by the oscillatory growth of a pressure perturbation.

1 Introduction

If two elastic bodies at different temperatures are pressed together, there will generally be some resistance to heat flow across the interface, which will be a function of the local contact pressure (Cooper et al. (1969), Shlykov and Ganin (1964)). However, the contact pressure is itself influenced by the thermal distortion, and hence by the temperature field in the bodies (Clausing (1966), Barber (1973)), so the heat conduction and elasticity solutions are coupled through the boundary conditions. An interesting consequence is that neither uniqueness nor existence theorems can be proved in general for steady-state solutions of such coupled problems (Barber (1978)), though Duvaut (1979) has proved existence for a particular contact resistance/pressure law.

Thermomechanical coupling also raises questions of stability, which have been investigated for various one-dimensional systems, using perturbation methods (Barber et al. (1980), Barber and Zhang (1988)). If one of the bodies is rigid, it is found that the criteria for stability and uniqueness are identical—in other words, steady-state solutions are only unstable when they are nonunique, in which case there exists another stable solution to which the system will gravitate. However, when both bodies are deformable, stability depends additionally on the ratio of thermal diffusivities of the materials and the behavior is more complex. For example, in the contact of two one-dimensional rods, several qualitatively different stability patterns are obtained, depending on diffusivity ratios and the rod lengths (Barber and Zhang (1988)). In particular, this system can exhibit instability even when the steady-state solution is unique, in which case numerical studies show that a steady nonlinear oscillatory behavior is obtained.

Two and three-dimensional problems are considerably more challenging, but it has been shown that a sinusoidal contact pressure perturbation on a nominally uniform (one-dimensional) pressure between two half spaces will be unstable for sufficiently large heat fluxes (Barber (1987)). The solution follows a method developed by Dow and Burton (1972) for the analogous problem of thermoelastic instabilities in contact pressure due to frictional heating.

Contributed by the Applied Mechanics Division of The American Society of Mechanical Engineers for publication in the Journal of Applied Mechanics.

Discussion on this paper should should be addressed to the Technical Editor, Leon M. Keer, The Technological Institute, Northwestern University, Evanston, IL 60208, and will be accepted until two months after final publication of the paper itself in the Journal of Applied Mechanics. Manuscript received and accepted by the ASME Applied Mechanics Division, March 27, 1989.

2 Instability of a Sinusoidal Pressure Perturbation

Consider the one-dimensional problem of two half spaces of dissimilar materials in contact over their common interface, $y=0$, with a uniform contact pressure, p_0, and a uniform heat flux, $q_y=q_0$. We postulate the possibility of an exponentially growing small perturbation in the temperature field of the form

$$T(x,y,t)=f(y)e^{bt}\cos(mx) \tag{1}$$

where x is a spatial coordinate in the plane of the interface, t is time, and b, m are constants. The function $f(y)$ is determined from the requirement that the temperature must satisfy the transient heat conduction equation, after which the change in contact pressure due to thermal distortion can be calculated. The change in contact resistance, $R(p)$, due to this contact pressure perturbation can then be found. Finally, we deduce that a perturbation of the form (1) is possible if and only if the change in contact resistance is consistent with the corresponding thermal boundary conditions at the interface.

The analysis is given in (Barber (1987)) and will not be repeated here, but we note that the system is self-excited, and hence leads to a set of homogeneous equations which have a nontrivial solution only for certain eigenvalues of the exponential growth rate, b. The characteristic equation for the eigenvalues is found to be

$$4MR' q_0 m\left[\frac{\delta_1}{a_1(a_1+m)}+\frac{\delta_2}{a_2(m-a_2)}\right]$$
$$+R_0+\frac{1}{K_1 a_1}-\frac{1}{K_2 a_2}=0 \quad (2)$$

where

$$a_1=\sqrt{m^2+b/k_1};\ a_2=-\sqrt{m^2+b/k_2} \quad (3)$$

$$\frac{1}{2M}=\frac{1-\nu_1}{\mu_1}+\frac{1-\nu_2}{\mu_2} \quad (4)$$

$$\delta_i=\frac{\alpha_i(1+\nu_i)}{K_i}. \quad (5)$$

R_0, R' denote the contact resistance, $R(p_0)$, and the derivative, $dR(p)/dp$, at the pressure, p_0, respectively, and K_i, k_i, α_i, ν_i, μ_i, $i=1, 2$, are the thermal conductivity, diffusivity, coefficient of thermal expansion, Poisson's ratio, and modulus of rigidity, respectively, for the two materials. The physical property δ is generally referred to as the distortivity, since it relates the thermoelastic distortion to the local heat flux in steady-state two-dimensional problems (Dundurs (1974)).

Equation (2) might have many solutions, some real and some complex, each of which would correspond to a possible perturbation (eigenfunction) of the form (1). The original one-dimensional system will be unstable if *any* of these solutions for b is positive or complex with positive real part. In the present paper, we shall investigate in some detail the conditions under which this can happen and, in particular, how the stability behavior is influenced by the properties of the contacting materials.

3 Dimensionless Presentation

It is convenient to restate equation (2) in dimensionless form, by defining the following dimensionless parameters

$$R^*=R_0 K_1 m \quad (6)$$
$$Q^*=-4MR'\alpha_1(1+\nu_1)q_0 \quad (7)$$
$$z=b/(m^2 k_1) \quad (8)$$
$$r_1=k_2/k_1;\ r_2=\delta_2/\delta_1;\ r_3=K_2/K_1 \quad (9)$$
$$c_1=(1+z)^{1/2};\ c_2=(1+z/r_1)^{1/2}. \quad (10)$$

Notice that we have introduced a negative sign into the definition (7), since the contact resistance, $R(p)$, is generally a monotonically decreasing function of p, and hence R' is negative. The definition (7) will ensure that Q^* generally has the same sign as the heat flux q_0.

Substituting the expressions (6)–(10) into equation (2), we obtain

$$Q^*=\frac{R^*+\dfrac{1}{c_1}+\dfrac{1}{r_3 c_2}}{\left[\dfrac{1}{c_1(1+c_1)}-\dfrac{r_2}{c_2(1+c_2)}\right]}. \quad (11)$$

This equation cannot be solved for z in closed form, but we can solve it inversely for Q^* if z is given. We first consider the simpler case in which z is assumed to be real.

4 Real Roots

The function $Q^*(z)$ defined by equation (11) can take any of the four forms illustrated in Fig. 1(a–d), depending upon the ratios of material properties, r_1, r_2, r_3 and the dimensionless resistance, R^*. It is convenient to label the materials in such a way that $r_1>1$. This involves no loss in generality.

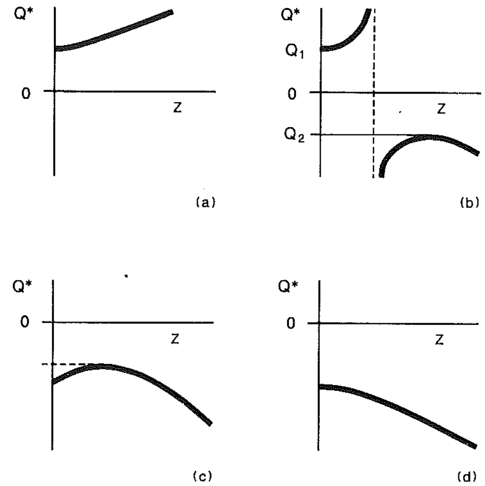

Fig. 1 Form of the function $Q^*(z)$ (equation (11)) for various ratios of material properties

If z is real and positive, it follows that c_1, c_2 are also real and positive and, hence, the numerator of the right-hand side of (11) is a positive decreasing function of z. The denominator of this expression can be written

$$D(z, r_1, r_2)=\frac{1}{c_1(1+c_1)}-\frac{r_2}{c_2(1+c_2)},$$
$$=f(z)-r_2 f(z/r_1) \quad (12)$$

where

$$f(x)=\frac{1}{\sqrt{1+x}(1+\sqrt{1+x})}=\frac{1}{x}-\frac{1}{x\sqrt{1+x}} \quad (13)$$

is a monotonically decreasing function of z, with limiting behavior

$$f(0)=1/2;\ f(x)\to 1/x \text{ as } x\to\infty. \quad (14)$$

The corresponding limiting behavior of $D(z, r_1, r_2)$ is therefore

$$D(0, r_1, r_2)=(1-r_2)/2 \quad (15)$$
$$D(z, r_1, r_2)\to(1-r_2 r_1)/z \text{ as } z\to\infty. \quad (16)$$

If a zero of equation (11) is to reach the real axis through the origin, the corresponding stability boundary will be determined by the condition $z=0$, i.e.,

$$Q^*=2(R^*+1+1/r_3)/(1-r_2) \quad (17)$$

which is possible only for the heat flow direction defined by $Q^*(1-r_2)>0$.

Recalling that $r_1>1$, we deduce immediately from equation (16) that at sufficiently large z, equation (11) will define values of Q^* of the opposite sign to (17), as in Fig. 1(b), provided that $1>r_2>1/r_1$. These zeros clearly cannot reach the real axis through the origin with increasing $|Q^*|$ and hence must pass into the right half plane across the imaginary axis. Material combinations whose property ratios fall into this range will exhibit instability for sufficiently large heat fluxes of *either* sign, the boundary for real roots being determined by equation (17) if $Q^*>0$ and by the *minimum* Q^* defined by (11) at larger z if $Q^*<0$. These values are labeled Q_1, Q_2, respectively, in Fig. 1(b).

Even for cases where a zero at the origin is possible, it can arise that the derivative of $|Q^*|$ with respect to z at the origin is negative as in Fig. 1(c), indicating that zeros occur on the positive real line at *lower* values of $|Q^*|$ than those needed to produce a zero at the origin. As before, these zeros must reach

Table 1 Type of behavior for different material combinations

Material No.																			
1																			
2	2																		
3	2	2																	
4	2	2	3																
5	2	2	2	1															
6	2	2	1	1	1														
7	1	3	2	2	3	2													
8	2	3	1	5	3	1	3												
9	1	2	2	2	2	2	1	2											
10	1	3	2	2	2	2	4	3	2										
11	1	2	2	2	2	2	1	2	1	2									
12	2	2	2	2	2	2	1	2	3	1	1								
13	2	2	2	2	2	2	1	2	4	1	1	1							
14	2	2	1	1	2	1	3	3	2	2	2	2	2						
15	1	2	2	2	2	2	1	2	1	1	1	1	1	2					
16	1	3	2	2	3	2	3	3	1	3	4	1	1	3	1				
17	1	1	4	5	1	3	1	5	1	1	1	1	1	5	1	1			
18	1	1	3	3	5	2	1	4	1	1	1	1	1	4	1	1	1		
19	1	3	2	2	2	2	2	3	1	2	3	1	1	2	1	1	1	1	
20	1	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

Material No.

Table 2 Representative properties of some structural materials

No.	Material	ν	α $°C^{-1} \times 10^{-6}$	K $W/m°C$	k mm^2/s	δ $(m/W) \times 10^{-6}$
1	Zirconia	0.26	10.8	2.2	0.81	6.185
2	Nodular cast iron	0.31	13.7	48.9	16.05	0.367
3	Aluminium alloy	0.32	22.0	173.0	67.16	0.168
4	Beryllium/copper	0.29	17.0	147.0	42.17	0.149
5	Brass/bronze	0.33	19.0	78.0	21.35	0.324
6	Copper	0.33	17.0	381.0	101.93	0.059
7	Iron(gray cast)	0.26	12.0	50.0	12.86	0.302
8	Magnesium alloy	0.35	26.0	95.0	45.11	0.369
9	Nickel alloy	0.30	13.0	21.0	5.06	0.805
10	Carbon steel	0.30	12.0	47.0	13.27	0.332
11	Alloy steel	0.30	11.0	38.0	10.73	0.376
12	Stainless steel	0.30	14.0	21.0	5.93	0.867
13	Titanium alloy	0.33	9.0	12.0	5.45	0.998
14	Zinc alloy	0.33	27.0	111.0	36.56	0.324
15	Partially stabilized Zirconia	0.23	10.6	1.8	0.78	7.243
16	Alumina	0.20	7.7	34.0	8.52	0.272
17	SiC sintered	0.16	4.4	110.0	35.48	0.046
18	SiN Hot pressed	0.28	2.6	33.0	18.75	0.101
19	SiN Reaction bonded	0.24	2.9	10.0	8.00	0.360
20	Granite	0.28	6.5	2.5	1.09	3.328

the right half-plane by crossing the imaginary axis as $|Q^*|$ increases. Recalling that the numerator in (11) is a monotonically decreasing function of z, we note that $|Q^*|$ will decrease with z at small z for all values of R^*, r_3, provided that $|D(z, r_1, r_2)|$ *increases* with z in the same range. This in turn depends upon the sign of D and its first nonzero derivative at $z=0$. Differentiating (12) and setting z to zero, we obtain

$$D'(0, r_1, r_2) = 3(r_2/r_1 - 1)/8 \tag{18}$$

which will have the same sign as (15) if $r_1 > r_2 > 1$. We therefore conclude that the stability boundary will always be determined by a zero on the imaginary axis (corresponding to a growing oscillatory solution) if $r_1 > r_2 > 1$.

So far, we have only considered the influence of the denominator of equation (11) on the slope at $z=0$. By differentiating (11) and deleting multipliers known to be positive, we can show that $Q'(0)$ has the same sign as the expression

$$F(R^*) = 3R^*(1 - r_2/r_1) + 3(1 + 1/r_3)(1 - r_2/r_1) - 2(1 - r_2)(1 + 1/(r_1 r_3)) \tag{19}$$

which can also be regrouped in the form

$$F(R^*) = 3R^*(1 - r_2/r_1) + \{2(1 - 1/r_1) + (1 - r_2/r_1)\} + 3r_2(1 - 1/r_1) + (1 - r_2). \tag{20}$$

If $r_2 < 1$, every term in (20) is positive, as is (15), indicating that Q^* increases as we move away from the origin along the real line, as in Fig. 1(*a*, *b*). Figure 1(*b*) corresponds to the range $1 > r_2 > 1/r_1$, Fig. 1(*a*) to $r_2 < 1/r_1$.

We have already demonstrated from the properties of D alone that $Q^*(0)$ and $Q^{*\prime}(0)$ are of opposite sign for $r_1 > r_2 > 1$ and it therefore remains to consider the range $r_2 > r_1$. In this range, $Q^*(0)$ is negative (see (17)) and $F(R^*)$ will be negative for all R^* provided

$$3(1 + 1/r_3)(r_2/r_1 - 1) - 2(r_2 - 1)(1 + 1/(r_1 r_3)) > 0 \tag{21}$$

giving the form illustrated in Fig. 1(*d*). However, if $r_2 > r_1$ but (21) is not satisfied, there will be a range of values of R^* near $R^* = 0$, for which $F(R^*)$ is positive, and hence Fig. 1(*c*) behavior is obtained.

5 Summary of Stability Behavior

The preceding results enable us to characterize the stability behavior of an interface between dissimilar bodies on the basis of the dimensionless ratios of material properties. Three distinct kinds of behavior can be identified:

(*a*) Instability occurs for heat flow into the more distortive material, the stability boundary being determined by the passage of a zero through the origin in the complex plane, the corresponding criterion from (17) being

$$Q^*(1 - r_2) > 2(R^* + 1 + 1/r_3). \tag{22}$$

(*b*) Instability occurs for heat flow into the more distortive material, but at a smaller heat flux than that determined by (22). In this case, initial instability is characterized by a complex root, corresponding to a growing oscillatory solution.

(*c*) Instability occurs for heat flow into the *less* distortive material. Initial instability again corresponds to a complex root.

Labeling the bodies such that $r_1 > 1$, we find that

(1) For $0 < r_2 < 1/r_1$, type (a) behavior is always obtained.

(2) For $1/r_1 < r_2 < 1$, type (a) behavior is obtained for heat flow into the more distortive material and type (c) behavior for the opposite direction.

(3) For $1 < r_2 < r_1$, type (b) behavior is always obtained.

(4) If $r_1 < r_2$ *and* (21), type (a) behavior is always obtained.

(5) If $r_1 < r_2$ and (21) is *not* satisfied, there will be a critical value of R^* given by $F(R^*) = 0$, (see equation (19)), *below* which the behavior is of type (b) and *above* which it is of type (a).

Any given combination of materials at the interface must fall into one of these categories and we list the behavior obtained for interfaces between a representative selection of materials in Table 1. The material properties used in the development of this Table are given in Table 2.

We note from Table 1 that material combinations of type (1) and type (2) are about equally probable and together account for 89 percent of those considered. This is a consequence of the strong correlation between thermal conductivity, diffusivity, and distortivity.

6 Complex roots

For material combinations exhibiting behavior of type (b) or (c), the stability boundary corresponds to the heat flux at which a zero of equation (11) first crosses the imaginary axis. We must therefore examine the properties of this equation for complex values of z.

It is convenient to consider the more general case in which equation (11) has a solution of the form

$$z = we^{i\theta} \tag{23}$$

where w, θ are real.

Defining

$$H(z, r_1, r_3) = 1/c_1 + 1/(r_3 c_2), \tag{24}$$

we can rewrite equation (11) in the form

$$Q^* D(z, r_1, r_2) - H(z, r_1, r_3) - R^* = 0. \tag{25}$$

Separating real and imaginary parts and noting that Q^* and R^* must be real, we obtain

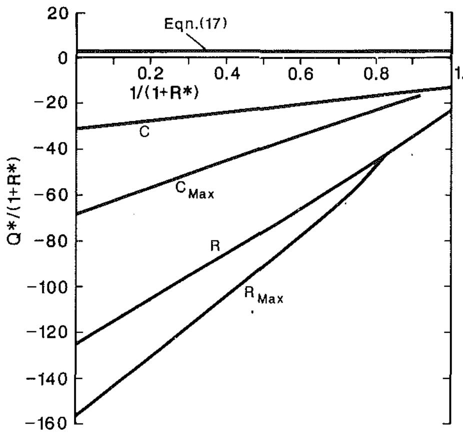

Fig. 2 Stability boundaries for an interface between aluminum alloy and stainless steel (type (2) behavior)

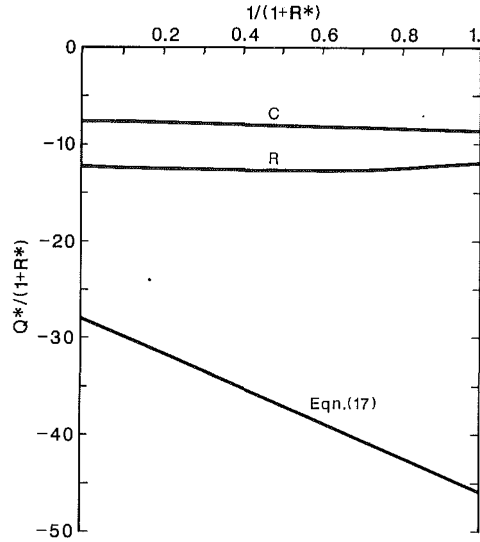

Fig. 3 Stability boundaries for an interface between cast iron and brass (type (3) behavior)

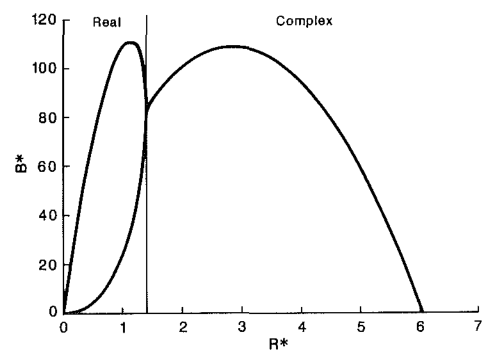

Fig. 4 Exponential growth rate of a perturbation as a function of wavelength for an aluminum alloy/stainless steel interface. Results are for a heat flux $Q^* = -200$.

$$Q^*\mathrm{Re}(D) - \mathrm{Re}(H) - R^* = 0 \tag{26}$$

$$Q^*\mathrm{Im}(D) - \mathrm{Im}(H) = 0 \tag{27}$$

which can be solved inversely for a given value of z in the form

$$Q^* = \mathrm{Im}(H)/\mathrm{Im}(D) \tag{28}$$

$$R^* = Q^*\mathrm{Re}(D) - \mathrm{Re}(H). \tag{29}$$

The relation between Q^* and R^* at the complex stability boundary is then defined parametrically by setting $\theta = \pi/2$ and allowing w to take values in the range $w > 0$, noting that only those results in which $R^* > 0$ correspond to physically admissible solutions. This method also has the advantage that by setting θ to values in the range $0 < \theta < \pi/2$, we can find the conditions for unstable solutions elsewhere in the complex plane and hence track the motion of the zeros of equation (11) through the plane as Q^* increases. In particular, we can verify that they reach the positive real axis at the point corresponding to the minimum Q^* for a real root determined from Section 4 of this paper.

7 Results

Figure 2 shows the stability boundaries obtained for an interface between aluminum alloy and stainless steel, which exhibits type (2) behavior (see Table 1). Since R^* can take any positive value, it is convenient to condense the infinite range by plotting $Q^*/(1+R^*)$ against the function $1/(R^*+1)$, which is always between 0 and 1. Two curves are obtained for negative Q^*, one defining the minimum Q^* for a zero on the positive real line (curve R) and one for a zero on the imaginary axis (curve C). Curve C, of course, defines the true stability boundary for this direction of heat flow.

For positive Q^*, the stability boundary is defined by (17) and always corresponds to a real root.

Figure 3 shows the corresponding results for an interface between cast iron and brass, which exhibits type (3) behavior. Instability only occurs for $Q^* < 0$, and curves are shown for imaginary (C) and real (R) zeros as before.

8 Dominant Wavelengths

The stability behavior just described shows considerable similarity to that obtained in the related problem where heat is generated at the interface due to sliding (Dow and Burton (1972)). Azarkhin and Barber (1985) developed a numerical solution to investigate the behavior of such a system and noted that an initially smooth contact pressure distribution tends to develop wavy perturbations of a well-defined spatial wavelength, corresponding to the maximum exponential growth rate in Dow and Burton's perturbation analysis (Dow and Burton (1972)).

We might anticipate a similar result for the present system. We note from equation (11) that instability occurs at a lower value of Q^* if R^* is small, corresponding to small values of m, and hence long wavelength perturbations. However, long wavelength perturbations involve changing the temperature of large masses of material and are likely to have low exponential growth rates.

We can investigate this question by using the inverse solution technique of Sections 4 and 6 iteratively to determine z for given values of Q^* and R^*. The results for an interface between aluminum alloy and stainless steel are presented in Fig. 4. We remove the wavelength dependence in the dimensionless parameter z by plotting the real part of the product

$$B^* = z(R^*)^2 = (R_0K_1)^2 b/k_1 \tag{30}$$

as a function of R^* for various values of Q^*. For given values of R_0 and the material properties, B^* is proportional to b and R^* is proportional to m. Thus, Fig. 4 shows the effect of m on the real part of b, which defines the rate at which the magnitude of the perturbation grows. When the roots are real, two distinct curves are obtained, since the zeros of equation (11) bifurcate when they reach the real line, one moving towards the origin and one towards infinity. Of course, the root with the larger growth rate will dominate the transient behavior in this case.

The curve shows two maxima for B^*—one in the complex root range and one on the upper branch of the root curve. The corresponding values of R^* are identified by the curves C_{max} and R_{max} in Fig. 2.

The two maximum values of B^* are relatively close, indicating that both wavelengths would play a significant part in the initial transient behavior of the system.

For the cast iron/brass interface, we find that the curves similar to Fig. 4 show a maximum at the complex/real transition, and hence the dominant wavelength corresponds to the curve R in Fig. 3.

9 Conclusions

An analysis of the stability criterion for the thermoelastic contact of two dissimilar materials shows that material combinations can be classified into one of five categories depending upon the ratios of the thermal conductivities, diffusivities, and distortivities.

The most common cases are:

(1) Stability behavior is similar to that for one material systems—i.e., the stability criterion is independent of the thermal diffusivities. In particular, instability only occurs when the heat flows into the material of higher distortivity, which is also the condition required for nonuniqueness of the steady-state solution.

(2) The same stability criterion is obtained for heat flow into the more distortive material, but instability is also possible for heat flow into the less distortive material. In the latter case, the steady-state solution is believed to be unique and the stability boundary is characterized by the growth of sinusoidal perturbations with complex growth rates, indicating oscillatory behavior.

When the system is unstable, the exponential growth rate depends upon the spatial wavelength of the perturbation and hence a range of preferred wavelengths is expected to dominate the transient behavior. This question is currently being investigated, using a direct numerical solution of the full nonlinear two-dimensional thermoelastic contact problem.

Acknowledgments

The authors are pleased to acknowledge the support of the National Science Foundation through grant number MSM-8719073.

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