Thermoelastic Green's Functions for Plane Problems in General Anisotropy

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1 Introduction

A major contribution to the study of general anisotropic elasticity was made by Eshelby et al. (1953), who showed that, by making an appropriate linear transformation of the coordinate axes, the governing equations of the plane problem could be reduced to Laplacian form, thus permitting solutions to be written down in terms of analytic functions of the complex variable.

This method was elaborated by Stroh (1958), who also gave the solutions for a plane dislocation and for a Griffith crack perturbing an otherwise uniform tensile stress.

Eshelby and Stroh's method was extended to plane thermoelasticity by Clements (1973). The results were used by Atkinson and Clements (1977) to treat the Griffith crack problem with heat flow.

2 Solutions Using Green's Functions

The above authors used Fourier transforms to represent the stress and displacement fields, but an alternative method, which has been extensively used in isotropic crack and contact problems, is to represent the elastic fields in terms of Green's functions (see, e.g., Comninou, 1977). Green's functions for isothermal problems have been discussed by Atkinson (1966) and Sinclair and Hirth (1975).

The Green's function method has some important advantages over transform methods. In particular, we note that the solution is expressed in terms of physical variables so that it is easier to determine at intermediate stages whether the solution is physically reasonable. Also, it is usually possible to express the solution in terms of distributions of Green's functions over a finite range, with bounded or integrable singular behavior at the end points. This ensures that the resulting integral equations will have regular solutions. In contrast, Fourier transform methods require that we pay very careful attention to the way the representation behaves at infinity, to avoid possible divergent integrals. This is particularly important in thermoelastic problems (Barber, 1983).

Green's functions have not been extensively used for thermoelastic problems, but appropriate functions for the frictionless contact of two isotropic half-planes are given by Dundurs and Comninou (1979a,b) and used by Comninou and Dundurs (1980) to treat a problem in which separation between such bodies occurs in a central region. It is the purpose of the present paper to develop the appropriate plane thermoelastic Green's functions for the generally anisotropic material.

3 The Temperature Field

The thermal analog of a dislocation is a temperature field which exhibits a constant discontinuity along the half-line $x_2=0$, $x_1>0$ in Cartesian coordinates x_1, x_2, x_3, i.e.,

$$T(x_1,0^+)=0; \quad x_1>0 \tag{1}$$

$$T(x_1,0^-)=T_0; \quad x_1>0 \tag{2}$$

and which involves no net source at the origin. A related isotropic Green's function was introduced by Dundurs and Comninou (1979a), who described it as a "heat vortex," so we shall retain this terminology here.

The temperature distribution, T, must satisfy the heat conduction equation

$$K_{ij}\frac{\partial^2 T}{\partial x_i \partial x_j}=0 \tag{3}$$

where $K_{ij}=K_{ji}$ are the coefficients of heat conduction. By a suitable linear transformation of coordinates, equation (3) can be reduced to plane harmonic form. We define the modified complex variable $z_t=x_1+\tau x_2$, where τ is the root with positive imaginary part of the equation

$$K_{11}+2K_{12}\tau+K_{22}\tau^2=0 \tag{4}$$

With this notation, it follows that any distribution of the form $T(x_1,x_2)=f(z_t)$ satisfies equation (3) where $f(z_t)$ is an analytic function of z_t.

A suitable function satisfying the boundary conditions (1) and (2) can be written as

$$T(x_1,x_2)=T_0(\log z_t-\log\bar{z}_t)/4\pi i \tag{5}$$

where $\bar{z}_t$ denotes the complex conjugate $x_1+\bar{\tau}x_2$. The heat flux is

$$q_i=-K_{ij}\frac{\partial T}{\partial x_j}=(-T_0/4\pi i)\{K_{i1}(1/z_t-1/\bar{z}_t)$$

$$+K_{i2}(\tau/z_t-\bar{\tau}/\bar{z}_t)\} \tag{6}$$

In particular, on the plane $x_2=0$ we have

$$q_2=-(T_0K_{22}/4\pi i)\{\tau-\bar{\tau}\}/x_1 \tag{7}$$

4 Particular Thermoelastic Solution

Following Clements (1973), we can develop a particular solution of the equilibrium equation in the form

$$u_k=C_kf(z_t)+\bar{C}_k\bar{f}(\bar{z}_t) \tag{8}$$

$$T=Af'(z_t)+\bar{A}\bar{f}'(\bar{z}_t) \tag{9}$$

where C_k are obtained from the equation

$$D_{ik}C_k=A\Gamma_i \tag{10}$$

with D_{ik} and Γ_i given by

$$D_{ik}=c_{i1k1}+\tau(c_{i1k2}+c_{i2k1})+\tau^2c_{i2k2} \tag{11}$$

$$\Gamma_i=\beta_{i1}+\tau\beta_{i2} \tag{12}$$

and c_{ijkl}, β_{ij} are the elastic constants and the stress-temperature coefficients, respectively. From equation (10) we have

$$C_k=AP_{ki}\Gamma_i \tag{13}$$

where

$$P_{ji}D_{ik}=\delta_{jk} \tag{14}$$

The stress components σ_{ij} can be obtained by substituting for u_k, T from equations (8) and (9) into the constitutive relation giving

$$\sigma_{ij}=A(N_{ij}-\beta_{ij})f'(z_t)+\bar{A}(\bar{N}_{ij}-\beta_{ij})\bar{f}'(\bar{z}_t) \tag{15}$$

(Clements, 1973), where

$$N_{ij}=c_{ijk1}P_{kl}\Gamma_l+\tau c_{ijk2}P_{kl}\Gamma_l \tag{16}$$

The temperature distribution of equation (5) can be expressed in the form of equation (9) by defining $A=T_0/4\pi i$ and

$$f(z_t)=z_t\log z_t-z_t \tag{17}$$

This function is continuous everywhere except on the real axis $x_2=0$, $x_1>0$ and hence u_k, σ_{ij} are continuous except on this line. We can, therefore, regard equation (17) as defining an

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appropriate particular solution for the temperature distribution (5), provided we make a cut on the real axis. The stresses σ_{ij} and displacements u_k then have different values on each side of the cut, the corresponding expressions being

$$u_k = \frac{T_0 E_k}{4\pi i}(x_1 \log x_1 - x_1)$$

$$-\frac{T_0 \bar{E}_k}{4\pi i}(x_1 \log x_1 - x_1);\ x_2 = 0^+,\ x_1 > 0 \tag{18}$$

$$= \frac{T_0 E_k}{4\pi i} x_1(\log x_1 - 1 + 2\pi i)$$

$$-\frac{T_0 \bar{E}_k}{4\pi i} x_1(\log x_1 - 1 - 2\pi i);\ x_2 = 0^-,\ x_1 > 0 \tag{19}$$

$$\sigma_{ij} = \frac{T_0}{4\pi i}(N_{ij} - \beta_{ij})\log x_1$$

$$-\frac{T_0}{4\pi i}(\bar{N}_{ij} - \beta_{ij})\log x_1;\ x_2 = 0^+,\ x_1 > 0 \tag{20}$$

$$= \frac{T_0}{4\pi i}(N_{ij} - \beta_{ij})(\log x_1 + 2\pi i)$$

$$-\frac{T_0}{4\pi i}(\bar{N}_{ij} - \beta_{ij})(\log x_1 - 2\pi i);\ x_2 = 0^-,\ x_1 > 0 \tag{21}$$

where

$$E_k = P_{ki}\Gamma_i \tag{22}$$

It follows that there is a discontinuity in displacement

$$\Delta u_k \equiv u_k(x_1, 0^+) - u_k(x_1, 0^-)$$

$$= -(E_k + \bar{E}_k)T_0 x_1/2;\ x_1 > 0 \tag{23}$$

and in stress

$$\Delta\sigma_{ij} \equiv \sigma_{ij}(x_1, 0^+) - \sigma_{ij}(x_1, 0^-)$$

$$= -\{(N_{ij} + \bar{N}_{ij})/2 - \beta_{ij}\}T_0 \tag{24}$$

From these equations we note that the stress discontinuity, $\Delta\sigma_{ij}$, is constant across the cut $x_2 = 0$, $x_1 > 0$, but the discontinuity in displacement varies linearly with x_1.

These discontinuities are unacceptable if the Green's function is to be used for the solution of problems involving one or more closed cracks in an infinite medium. We, therefore, seek a corrective isothermal solution for the cut plane which when superposed on the particular thermoelastic solution will restore continuity of stress and displacement across the cut.

5 The Isothermal Wedge Problem

The infinite body with a cut on the positive x_1 axis constitutes an infinite 360 deg wedge, defined by $0 < \theta < 2\pi$, $r > 0$ in polar coordinates (see Fig. 1). The corrective solution is required to satisfy boundary conditions which are equal and opposite to equations (23) and (24).

To solve this problem, we make use of the solution due to Stroh (1958), in which the displacement is written

$$u_k = \sum_\alpha A_{k\alpha}\phi_\alpha(z_\alpha) + \bar{A}_{k\alpha}\bar{\phi}_\alpha(\bar{z}_\alpha) \tag{25}$$

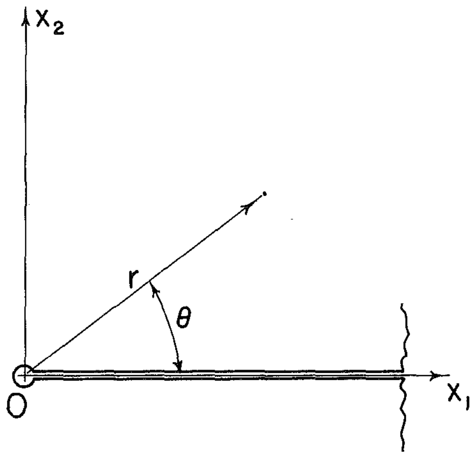
where

$$z_\alpha = x_1 + p_\alpha x_2 \tag{26}$$

and the p_α are the three roots with positive imaginary part of the equation

$$|c_{i1k1} + p_\alpha(c_{i1k2} + c_{i2k1}) + p_\alpha^2 c_{i2k2}| = 0 \tag{27}$$

The stress components σ_{ij} are defined in terms of the function



$$\Omega_i = \sum_\alpha L_{i\alpha}\phi_\alpha(z_\alpha) + \bar{L}_{i\alpha}\bar{\phi}_\alpha(\bar{z}_\alpha) \tag{28}$$

(Stroh, 1958), where

$$L_{i\alpha} = (c_{i2k1} + p_\alpha c_{i2k2})A_{k\alpha} \tag{29}$$

in particular, we have

$$\sigma_{i2} = \frac{\partial\Omega_i}{\partial x_1} = \sum_\alpha L_{i\alpha}\phi'_\alpha(z_\alpha) + \sum_\alpha \bar{L}_{i\alpha}\bar{\phi}'_\alpha(\bar{z}_\alpha) \tag{30}$$

For the 360 deg wedge with uniform tractions on the faces, we take.

$$\phi_\alpha = D_\alpha[z_\alpha \log z_\alpha - z_\alpha] \tag{31}$$

where D_α is a set of three complex constants which can be chosen to satisfy the six boundary conditions. Substituting into equations (25) and (30), we find

$$u_k = \sum_\alpha A_{k\alpha}D_\alpha x_1(\log x_1 - 1)$$

$$+ \bar{A}_{k\alpha}\bar{D}_\alpha x_1(\log x_1 - 1);\ x_2 = 0^+,\ x_1 > 0 \tag{32}$$

$$= \sum_\alpha A_{k\alpha}D_\alpha x_1(\log x_1 - 1 + 2\pi i)$$

$$+ \bar{A}_{k\alpha}\bar{D}_\alpha x_1(\log x_1 - 1 - 2\pi i);\ x_2 = 0^-,\ x_1 > 0 \tag{33}$$

$$\sigma_{i2} = \sum_\alpha L_{i\alpha}D_\alpha \log x_1 + \bar{L}_{i\alpha}\bar{D}_\alpha \log x_1;$$

$$x_2 = 0^+,\ x_1 > 0 \tag{34}$$

$$= \sum_\alpha L_{i\alpha}D_\alpha(\log x_1 + 2\pi i) + \bar{L}_{i\alpha}\bar{D}_\alpha(\log x_1 - 2\pi i);$$

$$x_2 = 0^-,\ x_1 > 0 \tag{35}$$

We conclude that this solution will cancel the discontinuity of equations (23) and (24) provided D_α satisfies the equations

$$2\pi i\sum_\alpha (A_{k\alpha}D_\alpha - \bar{A}_{k\alpha}\bar{D}_\alpha) = -T_0(E_k + \bar{E}_k)/2;\ k = 1,2,3 \tag{36}$$

$$2\pi i\sum_\alpha (L_{k\alpha}D_\alpha - \bar{L}_{k\alpha}\bar{D}_\alpha)$$

$$= -T_0\{(N_{k2} + \bar{N}_{k2})/2 - \beta_{k2}\};\ k = 1,2,3 \tag{37}$$

Equations (36) and (37) can be solved, giving

$$D_\alpha = -(T_0/4\pi i)\,\frac{\{(L_{k\alpha}(E_k+\bar{E}_k)/2+A_{k\alpha}[(N_{k2}+\bar{N}_{k2})/2-\beta_{k2}]\}}{L_{j\alpha}A_{j\alpha}} \tag{38}$$

Where we have used the following identities

$$L_{i\alpha}\bar{A}_{i\beta}+\bar{L}_{i\beta}A_{i\alpha}=0 \tag{39}$$

$$L_{i\alpha}A_{i\beta}+L_{i\beta}A_{i\alpha}=2L_{i\alpha}A_{i\alpha}\delta_{\alpha\beta} \tag{40}$$

Equation (39) was proved by Stroh (1958, p. 628), and equation (40) can be proven using a similar procedure.

6 The Thermoelastic Green's Function

The required thermoelastic Green's function can now be obtained by superposing the particular solution of Section 4 and the corrective solution of Section 5. In particular, we find that the stress on the plane $x_2=0$ is given by

$$\sigma_{i2}=T_0G_i\log|x_1| \tag{41}$$

where

$$G_i=\frac{1}{4\pi i}\left\{\sum_\alpha\left(\frac{L_{i\alpha}A_{k\alpha}}{L_{j\alpha}A_{j\alpha}}-\frac{\bar{L}_{i\alpha}\bar{A}_{k\alpha}}{\bar{L}_{j\alpha}\bar{A}_{j\alpha}}\right)\left(\frac{N_{k2}+\bar{N}_{k2}}{2}-\beta_{k2}\right)+\sum_\alpha\left(\frac{L_{i\alpha}L_{k\alpha}}{L_{j\alpha}A_{j\alpha}}-\frac{\bar{L}_{i\alpha}\bar{L}_{k\alpha}}{\bar{L}_{j\alpha}\bar{A}_{j\alpha}}\right)\left(\frac{E_k+\bar{E}_k}{2}\right)+N_{i2}-\bar{N}_{i2}\right\} \tag{42}$$

More general thermoelastic stress fields can be constructed by distributing Green's functions of the above form with an appropriate weight function. This method is particularly useful for representing the perturbation in the thermoelastic stress field in an infinite plane due to a closed crack. Green's functions are distributed along the line of the crack and the boundary conditions lead to integral equations for the unknown weight function. The problem of a plane crack in an infinite anisotropic plane is considered in a companion paper (Sturla and Barber, 1987).

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