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Disturbance at a Frictional Interface Caused by a Plane Elastic Pulse

We consider a plane pulse striking the frictional interface between two elastic solids which are held together by compressive applied tractions and sheared. The pulse causes a disturbance involving separation or slip between the bodies, which propagates along the interface at supersonic speed. The extent of these zones is determined using a convenient graphical representation and the interface tractions are given in closed form. It is found that the results change qualitatively when the coefficient of friction exceeds a critical value.

Introduction

The interface between two bodies in unbonded elastic contact exhibits an asymmetric behavior with respect to tensile and compressive tractions, the latter being physically admissible while the former are not. Such an interface is described as "unilateral" in contrast to the "bilateral" bonded interface which can transmit normal tractions of either sign.

The interaction of a plane elastic wave with a unilateral interface has been discussed in a number of recent papers [1-3]. The results can be obtained in closed form if the angle of incidence of the wave front is such that the disturbance propagates along the interface at a speed that is supersonic with respect to the materials of both bodies (i.e., if none of the reflected or refracted waves become surface waves). For this case, solutions have been given for an incident P or SV wave of harmonic form both with and without friction at the interface [1, 2] and the results for the frictionless interface were extended to a wave of arbitrary form in [3]. The work of Miller and Tran aimed at developing approximate methods for treating more general friction laws may also be noted [4].

In this paper we consider the problem of a wave of arbitrary form incident on an interface with Coulomb friction. We assume that the static and kinetic coefficients of friction are equal. In general, we anticipate the development of regions of slip and separation at the interface and a major part of the problem is to determine the extents of these regions from the controlling inequalities which are:

(*a*) The gap must be non-negative – i.e., there is no interpenetration of material.

(*b*) Normal tractions must be compressive.

(*c*) Tangential tractions must not exceed the limiting value at which slip occurs.

(*d*) Relative slip must be in the direction opposed by the tangential tractions – i.e., negative work is done by these tractions during slip.

A simple method will be developed for determining these regions and the normal and tangential tractions at the interface. It will be shown that the behavior of the interface changes qualitatively when the coefficient of friction exceeds a certain value that depends on the elastic constants.

Formulation and Method of Solution

We consider two half spaces of different materials pressed together and sheared by tractions $p\infty$, $q\infty$ applied at infinity as shown in Fig. 1. We require $|q\infty| < fp\infty$ to rule out the possibility of catastrophic slip. Now suppose that a plane elastic stress pulse with velocity c_0 strikes the interface at an angle of incidence θ_0. The disturbance due to the incident pulse will propagate along the interface with velocity

$$v = c_0/\sin\theta_0 \tag{1}$$

and we restrict attention to the case where v is supersonic with respect to both half spaces. The disturbance will therefore be stationary with respect to the dimensionless moving coordinate

$$\eta = k_0(x_1 \sin\theta_0 - c_0 t) \tag{2}$$

where the wave number k_0 can here be regarded as the reciprocal of a characteristic length for the pulse.

Following the notation of the previous papers we denote the velocity of propagation of P and SV waves in the lower body by c_L, c_T, respectively, and use bars to distinguish the corresponding quantities for the upper body. The angles of reflection and refraction θ_i ($i = 1, 2, 3, 4$) are illustrated in Fig. 1 and are related by the equation

$$\frac{\sin\theta_0}{c_0} = \frac{\sin\theta_1}{c_L} = \frac{\sin\theta_2}{c_T} = \frac{\sin\theta_3}{\bar{c}_L} = \frac{\sin\theta_4}{\bar{c}_T} \tag{3}$$

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The Bilateral Solution. We first consider the bilateral problem in which the incident pulse strikes a *bonded* interface. The solution of this problem is algebraically tedious but routine and will not be given here. Derivations for the related case of a harmonic wave can be found in most books on elastic waves – e.g., [5, 6] and the case of an incident pulse of arbitrary form can be treated in the same way or by superposition using Fourier integrals.

For the supersonic case treated here, all the reflected and refracted pulses in the bilateral solution have the same wave form as the incident pulse and hence the tractions transmitted by the interface can be written in the form

$$\sigma_{22} = -p\infty + \mathcal{A}F(\eta) \equiv N_0(\eta) \tag{4}$$

$$\sigma_{12} = q\infty + \mathcal{B}F(\eta) \equiv S_0(\eta) \tag{5}$$

where $F(\eta)$ is determined by the shape of the pulse [3] and $\mathcal{A}$, $\mathcal{B}$ are constants depending on the material properties and angle of incidence.

The Corrective Solution. The values of N_0, S_0 calculated for the bilateral problem may violate the physical conditions given in the Introduction in some region, in which case the bilateral and unilateral solutions will differ, separation or slip regions occurring in the latter. Note however that these regions do not necessarily coincide with the regions of violation in the bilateral solution.

To treat this case, we develop a corrective solution which is superposed on the bilateral solution to give the unilateral solution.

In the frictionless case [3], such a solution was obtained from results for a moving dislocation at the interface. The same method could be used here, but it proves to be algebraically more efficient to use the results for a force pair moving along the interface.

We first consider the lower body alone, with a tangential force Q and a normal (tensile) force P acting at a point O moving to the right at supersonic velocity v over the surface as shown in Fig. 2.

The solution is given by Eringen and Suhubi [7] as follows

$$\frac{\partial u_1}{\partial \eta} = \{-m_2(1+m_2^2)Q - (1+2m_1m_2 - m_2^2)P\}\frac{\delta(\eta)}{\mu R} \tag{6}$$

$$\frac{\partial u_2}{\partial \eta} = \{(1+2m_1m_2 - m_2^2)Q - m_1(1+m_2^2)P\}\frac{\delta(\eta)}{\mu R} \tag{7}$$

where u_1 and u_2 are the surface displacements, $\delta(\eta)$ is the Dirac delta function and

$$m_1 = \left(\frac{v^2}{{c_L}^2} - 1\right)^{1/2} = \cot\theta_1, \tag{8}$$

$$m_2 = \left(\frac{v^2}{{c_T}^2} - 1\right)^{1/2} = \cot\theta_2, \tag{9}$$

$$R = (1-m_2^2)^2 + 4m_1m_2 \tag{10}$$

We now apply equal and opposite forces to the upper body (see Fig. 2) producing

$$\frac{\partial \bar{u}_1}{\partial \eta} = \{\bar{m}_2(1+\bar{m}_2^2)Q - (1+2\bar{m}_1\bar{m}_2 - \bar{m}_2^2)P\}\frac{\delta(\eta)}{\bar{\mu}\bar{R}} \tag{11}$$

$$\frac{\partial \bar{u}_2}{\partial \eta} = \{(1+2\bar{m}_1\bar{m}_2 - \bar{m}_2^2)Q + \bar{m}_1(1+\bar{m}_2^2)P\}\frac{\delta(\eta)}{\bar{\mu}\bar{R}} \tag{12}$$

where

$$\bar{m}_1 = \cot\theta_3, \quad \bar{m}_2 = \cot\theta_4, \quad \bar{R} = (1-\bar{m}_2^2)^2 + 4\bar{m}_1\bar{m}_2 \tag{13}$$

The force pair generates a gap

$$g(\eta) = \bar{u}_2 - u_2 \tag{14}$$

and a tangential shift

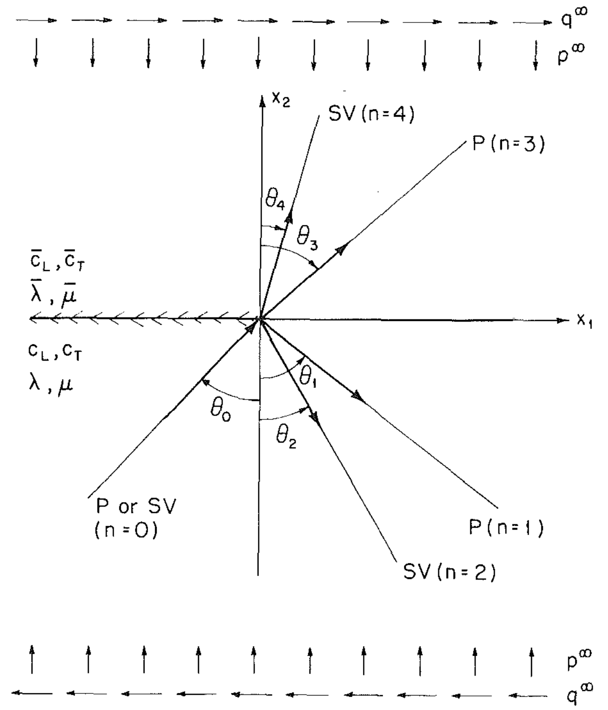

Fig. 1 Incident ($n = 0$), reflected ($n = 1, 2$), and refracted ($n = 3, 4$) waves

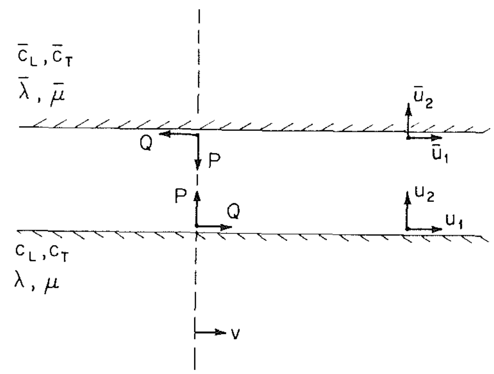

Fig. 2 Force pair moving at the interface with speed v. The two bodies are shown separated for clarity.

$$h(\eta) = \bar{u}_1 - u_1 \tag{15}$$

Note that a positive value of h corresponds to the upper body slipping to the right over the lower body.

Substituting from (6), (7), (11), (12) into (14) and (15), we have

$$\frac{\partial g}{\partial \eta} = \frac{\sin\theta_1}{\mu}(\lambda_2 P - \lambda_1 Q)\delta(\eta) \tag{16}$$

$$\frac{\partial h}{\partial \eta} = \frac{\sin\theta_1}{\mu}(\lambda_3 Q + \lambda_1 P)\delta(\eta) \tag{17}$$

where

$$\lambda_1 = \frac{\mu}{\sin\theta_1}\left[\frac{1+2m_1m_2 - m_2^2}{\mu R} - \frac{1+2\bar{m}_1\bar{m}_2 - \bar{m}_2^2}{\bar{\mu}\bar{R}}\right], \tag{18}$$

$$\lambda_2 = \frac{\mu}{\sin\theta_1}\left[\frac{m_1(1+m_2^2)}{\mu R} + \frac{\bar{m}_1(1+\bar{m}_2^2)}{\bar{\mu}\bar{R}}\right], \tag{19}$$

$$\lambda_3 = \frac{\mu}{\sin\theta_1}\left[\frac{m_2(1+m_2^2)}{\mu R} + \frac{\bar{m}_2(1+\bar{m}_2^2)}{\bar{\mu}\bar{R}}\right] \tag{20}$$

The dimensionless coefficients λ_1, λ_2, λ_3 also arise in the solution for an incident harmonic wave [2]. We note that λ_2, $\lambda_3 > 0$, but λ_1 may be of either sign, and vanishes for identical materials.

Equations (16) and (17) can also be cast in terms of the gap-opening velocity and slip velocity using equation (2). Thus,

$$\dot{g}(\eta)=\frac{\partial g}{\partial \eta}\frac{\partial \eta}{\partial t}=-\frac{k_0 c_0 \sin\theta_1}{\mu}(\lambda_2 P-\lambda_1 Q)\delta(\eta), \tag{21}$$

$$\dot{h}(\eta)=-\frac{k_0 c_0 \sin\theta_1}{\mu}(\lambda_3 Q+\lambda_1 P)\delta(\eta) \tag{22}$$

The delta functions in these equations show that the force system produces a purely local effect. Hence, with a more general distribution of tractions $S_1(\eta)$, $N_1(\eta)$ at the interface, $\dot{g}$, $\dot{h}$ will depend only on the local tractions of the corrective solution.

Boundary Conditions

In the unilateral solution, the interface may contain regions of stick, slip, or separation, in each of which two boundary conditions must be imposed. In addition, one or more *inequalities must be satisfied corresponding to the physical* conditions listed in the Introduction. We first consider the equality conditions.

The solution is obtained by superposing the bilateral and corrective solutions and hence the total tangential and normal tractions are

$$S(\eta)=S_0(\eta)+S_1(\eta) \tag{23}$$

$$N(\eta)=N_0(\eta)+N_1(\eta) \tag{24}$$

The bilateral solution by definition involves no slip or separation and hence the unilateral values of $\dot{g}$, $\dot{h}$ are identical with those of the corrective solution.

Equalities.

Stick. In stick zones, we have $\dot{h}=0$, $\dot{g}=0$ and hence from equations (21) and (22), $S_1=0$, $N_1=0$. In other words, the bilateral tractions are unchanged

$$S=S_0, \quad N=N_0 \tag{25}$$

Slip. In slip zones, we must have $g=0$ and hence $\dot{g}=0$,

$$\lambda_2 N_1-\lambda_1 S_1=0 \tag{26}$$

from equation (21).

The second boundary condition is

$$S=-fN sgn \dot{h} \tag{27}$$

since N must be negative. We define *conforming* slip as that for which $\lambda_1 \dot{h}>0$ and hence

$$S_0+S_1=-f(N_0+N_1)sgn\lambda_1 \tag{28}$$

Solving (26) and (28) for S_1, N_1, we find

$$S_1=-\frac{\lambda_2(S_0+fN_0 sgn\lambda_1)}{\lambda_2+|\lambda_1|f} \tag{29}$$

$$N_1=-\frac{\lambda_1(S_0+fN_0 sgn\lambda_1)}{\lambda_2+|\lambda_1|f} \tag{30}$$

and hence the slip velocity is

$$\dot{h}=\frac{c_L(\lambda_1^2+\lambda_2\lambda_3)(S_0+fN_0 sgn\lambda_1)}{\mu(\lambda_2+|\lambda_1|f)} \tag{31}$$

from equation (22).

The total tractions are

$$S=S_0+S_1=\frac{f sgn\lambda_1(\lambda_1 S_0-\lambda_2 N_0)}{\lambda_2+|\lambda_1|f} \tag{32}$$

$$N=N_0+N_1=-\frac{\lambda_1 S_0-\lambda_2 N_0}{\lambda_2+|\lambda_1|f} \tag{33}$$

In *nonconforming* slip $\lambda_1 \dot{h}<0$, giving a change of sign in equation (27). A similar process gives

$$S=-\frac{f sgn\lambda_1(\lambda_1 S_0-\lambda_2 N_0)}{\lambda_2-|\lambda_1|f} \tag{34}$$

$$N=-\frac{\lambda_1 S_0-\lambda_2 N_0}{\lambda_2-|\lambda_1|f} \tag{35}$$

$$\dot{h}=\frac{c_L(\lambda_1^2+\lambda_2\lambda_3)(S_0-fN_0 sgn\lambda_1)}{\mu(\lambda_2-|\lambda_1|f)} \tag{36}$$

Separation. In separation zones, the tractions S, N are zero and hence from (23) and (24)

$$S_1=-S_0, \quad N_1=-N_0 \tag{37}$$

It follows from equations (21), (22) that

$$\dot{g}=\frac{c_L}{\mu}(\lambda_2 N_0-\lambda_1 S_0) \tag{38}$$

$$\dot{h}=\frac{c_L}{\mu}(\lambda_3 S_0+\lambda_1 N_0) \tag{39}$$

Inequalities. The physical conditions leading to inequalities serve to determine the extents of the various zones.

Stick. *In stick zones, we require that the normal tractions* be nontensile and the shear tractions do not exceed the value at slip, i.e.,

$$N\leq 0, \tag{40}$$

$$|S|\leq -fN \tag{41}$$

The condition (41) includes (40) and we have already shown that in stick zones the bilateral tractions are unchanged. Hence stick is possible if and only if

$$-fN_0\geq S_0\geq fN_0 \tag{42}$$

Conforming Slip. In conforming slip we still require nontensile normal tractions and hence

$$\lambda_1 S_0-\lambda_2 N_0\geq 0 \tag{43}$$

since

$$\lambda_2+|\lambda_1|f>0 \tag{44}$$

It is convenient to define the ratio

$$\hat{f}=\frac{\lambda_2}{|\lambda_1|} \tag{45}$$

in terms of which (43) can be written

$$S_0 sgn\lambda_1\geq \hat{f}N_0 \tag{46}$$

Furthermore, the definition of conforming slip requires $\lambda_1\dot{h}>0$ and hence from (31), (44),

$$S_0 sgn\lambda_1>-fN_0 \tag{47}$$

since

$$\lambda_1^2+\lambda_2\lambda_3>0 \tag{48}$$

Notice that both conditions (46) and (47) must be satisfied in a conforming slip zone. Clearly (46) is the stronger if $N_0>0$ and (47) if $N_0<0$, since $f, \hat{f}>0$.

Nonconforming Slip. In nonconforming slip, the expressions (34)–(36) for S, N, and $\dot{h}$ all involve the multiplier $(\lambda_2-|\lambda_1|f)$ which can be of either sign. We therefore consider the two cases separately.

(*a*) $f<\hat{f}$

If $f<\hat{f}$ (i.e., $\lambda_2>|\lambda_1|f$), condition (40) gives

$$S_0 sgn\lambda_1\geq \hat{f}N_0 \tag{49}$$

as in conforming slip, but we now need $\lambda_1\dot{h}<0$ which implies

$$S_0 sgn\,\lambda_1<fN_0 \tag{50}$$

using (36).

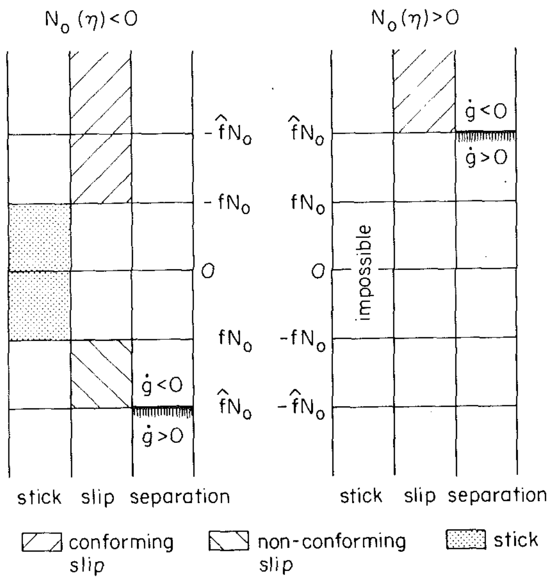

Fig. 3 Permissible ranges of $S_0(\eta)\,sgn\lambda_1$ for stick, slip, or gap for $f<\hat{f}$

Combining (49) and (50) we have

$$fN_0 > S_0 sgn\lambda_1 \geq \hat{f}N_0 \tag{51}$$

and hence nonconforming slip is only possible if $N_0 \leq 0$, since $f<\hat{f}$.

(*b*) $f>\hat{f}$

Applying similar arguments to the case $f>\hat{f}$, we find

$$fN_0 < S_0 sgn\lambda_1 \leq \hat{f}N_0 \tag{52}$$

Once again, nonconforming slip is only possible for $N_0 \leq 0$.

Separation. In the separation zone, we require that the gap $g \geq 0$. The equality conditions give an expression for $\dot{g}$ only, and hence we cannot deduce unique conditions on S_0, N_0 to be satisfied throughout the separation zone.

However it may be possible to determine the point at which separation starts, since the crack must then have a positive opening velocity ($\dot{g} \geq 0$) and hence a negative slope ($dg/d\eta \leq 0$). From equation (38) this implies

$$S_0 sgn\lambda_1 \leq \hat{f}N_0 \tag{53}$$

At the other end of the zone where the gap is closing we must have

$$S_0 sgn\lambda_1 \geq \hat{f}N_0 \tag{54}$$

If one of these two points can be determined uniquely, the other can always be found from the condition

$$\int_L \frac{dg}{d\eta}\, d\eta = 0 \tag{55}$$

where L is the extent of the separation zone.

Graphical Representation. The inequality conditions developed in the foregoing are all expressed in terms of the relationship between $S_0(\eta)sgn\lambda_1$ and $N_0(\eta)$ and can conveniently be summarized graphically.

(*a*) $f<\hat{f}$

We first consider the case $f<\hat{f}$ illustrated in Fig. 3. The diagrams show the ranges of the function $S_0 sgn\lambda_1$ for which stick, conforming, or nonconforming slip or gap are permitted. For example, stick is permitted only in the range

$$fN_0 \leq S_0 sgn\lambda_1 \leq -fN_0 \quad \text{for} \quad N_0<0 \tag{56}$$

These states cover all possible values of $S_0 sgn\lambda_1$ and are mutually exclusive, so we can uniquely define the state at any given point on the interface. The procedure is best explained

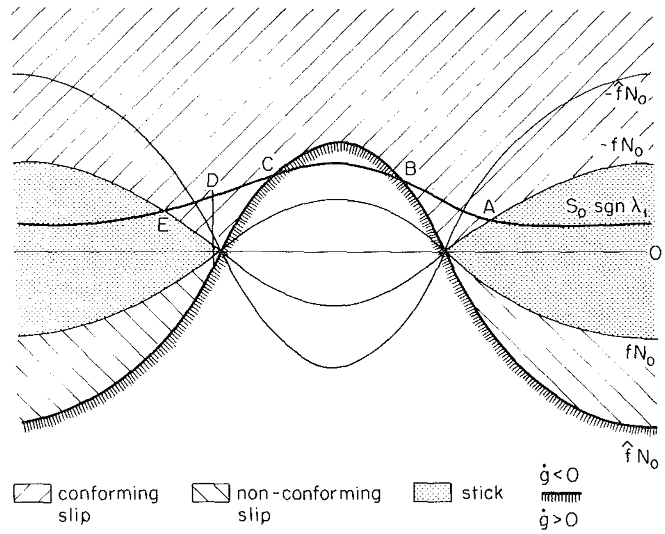

Fig. 4 Graphical determination of slip and separation zones for a typical example, $f<\hat{f}$

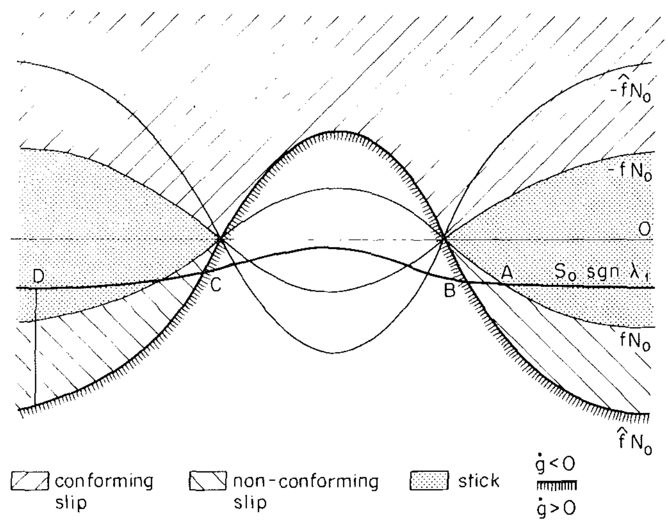

Fig. 5 Second example for $f<\hat{f}$ exhibiting a transition from separation to stick at D

through an example. When the bilateral problem is solved, we find N_0 as a function of η and hence plot the boundaries $\pm fN_0$, $\pm\hat{f}N_0$ as shown in Fig. 4. Notice that N_0 must tend to $-p\infty$ away from the pulse, but otherwise the shape chosen has no particular significance except as constrained by (4) and (5).

We next plot the value of $S_0 sgn\lambda_1$ on the same graph. Away from the pulse, this function tends to $q\infty\, sgn\lambda_1$ which must be between $\pm fN_0$. To the right of point A in Fig. 4, the interface must stick, while between A and B only conforming slip is possible. To the left of B, a separation zone is developed. The gap increases from B to C where $S_0 sgn\lambda_1$ is below the line $\hat{f}N_0$ and then starts to shrink. The closure point, D, is found from the condition that the gap there is zero. Thus, we choose D such that the algebraic sum of the areas between the lines $S_0 sgn\lambda_1$ and $\hat{f}N_0$ is zero. The location of D shows that in this case the interface passes from separation to conforming slip at the closure point and the interface sticks again at E.

A second example is illustrated in Fig. 5. There, nonconforming slip is developed in AB and the closure point D lies in the stick zone, indicating a direct transition from separation to stick. Note, however, that a direct transition from stick to separation is not possible.

It is clear from these examples that the controlling inequalities enable the arrangement of zones at the interface

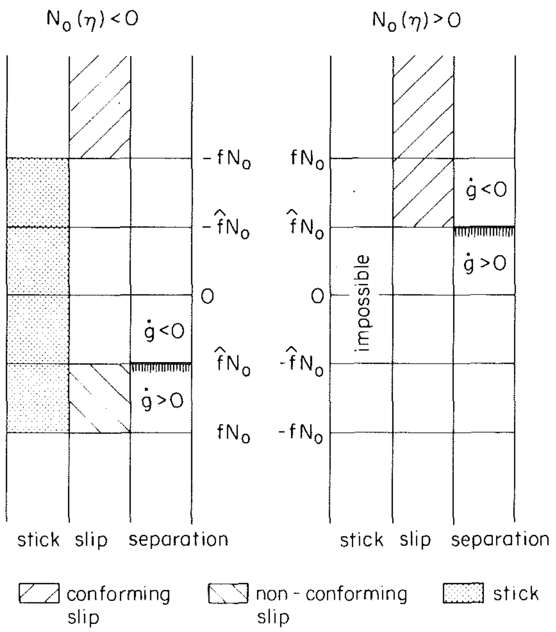

Fig. 6 Permissible ranges of $S_0(\eta)\, sgn\lambda_1$ for stick, slip, or gap, $f>\hat{f}$

to be determined simply and uniquely once the bilateral solution is known. The appropriate tractions and displacement velocities at the interface can then be written down using equations (23)–(39).

(*b*) $f>\hat{f}$

Corresponding results for the case $f>\hat{f}$ are shown in Figs. 6 and 7. From Fig. 6, we note that the ranges defined by the inequalities are not now mutually exclusive: three ranges overlap in $\hat{f}N_0 \geq S_0 sgn\lambda_1 \geq fN_0$ (which only exists for $N_0<0$). This suggests that certain problems may not have unique solutions. For the example shown in Fig. 7, stick must occur to the right of A, but the conditions in AB could be stick, nonconforming slip, or separation. Furthermore, no inconsistency arises in the regions to the left of B, whichever of these states is assumed.

We notice from equation (38) that the gap will start to open with a nonzero velocity unless the separation zone starts at A where $S_0 sgn\lambda_1 = \hat{f}N_0$. The condition that the gap opens smoothly can therefore be used to impose uniqueness on the problem and it has the effect of permitting only separation in the overlapping range.

We note, however, that no physical principles are violated by a velocity jump at the transition to separation, and indeed a jump in tangential velocity is implied by the expression for $\dot{h}$, (39), whatever conditions are assumed between A and B (for $f<\hat{f}$, continuity of both $\dot{g}$ and $\dot{h}$ is automatically satisfied at the transitions from stick to slip and slip to separation if the incident pulse has no step changes). An alternative hypothesis is that stick, once established continues until the inequalities make it inadmissible, i.e., that the transition from stick to separation occurs at B in Fig. 7.

A second paradoxical result for the case $f>\hat{f}$ is illustrated by the example in Fig. 8. If $q\infty$ lies in the range $-fp\infty < q\infty < -\hat{f}p\infty$, it is possible that a gap opened by the pulse never closes, since $S_0 sgn\lambda_1$ need never pass above the line $\hat{f}N_0$. The solids will then be separated to infinity on the left and are "unzipped" by the pulse, despite the presence of the compressive traction $p\infty$. This peculiar result can be traced back to the results for the moving force pair, equations (21) and (22). If P and Q are positive there is a limiting ratio of Q/P equal to $\hat{f}$ which, if exceeded, causes the force pair to leave a gap behind it, although the normal component P is tensile.

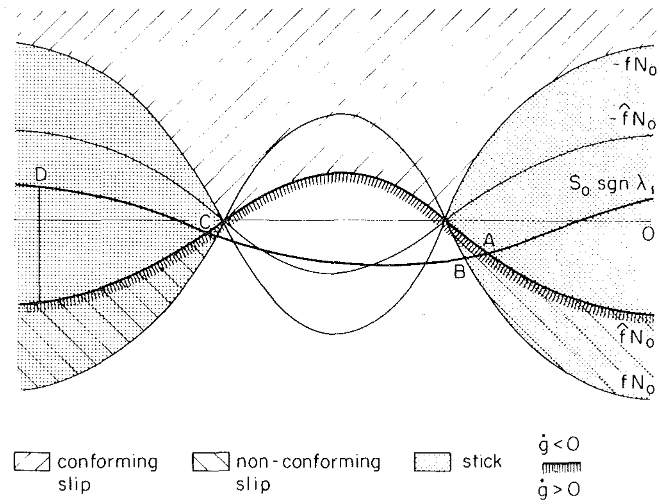

Fig. 7 Determination of slip and separation zones for $f>\hat{f}$

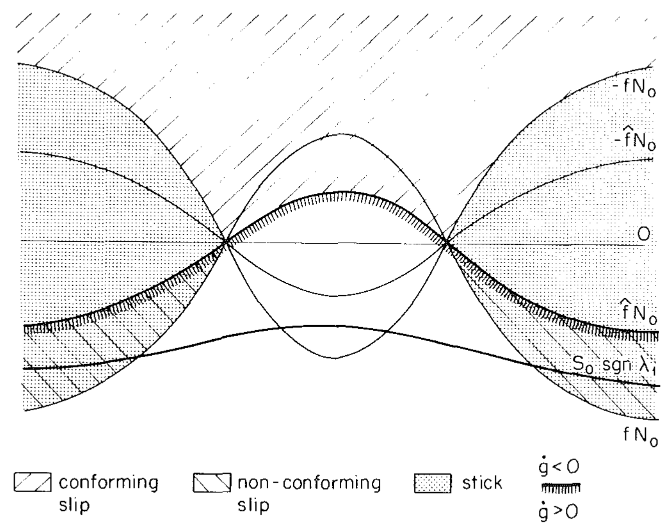

Fig. 8 Example for $f>\hat{f}$ in which the gap cannot close

In view of these various results, it is relevant to ask whether the condition $f>\hat{f}$ corresponds to any realistic combination of material properties. For similar materials, λ_1 is zero and hence $\hat{f}$ is infinite. Values of $\hat{f}$ that might be exceeded by a realistic coefficient of friction, can be obtained by choosing materials with significantly differing elastic moduli and an angle of incidence such that the largest of θ_i is close to $\pi/2$.

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