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Planar Hertz Contact With Heat Conduction

The paper discusses the planar Hertz contact problem when the bodies are not only pressed together but also exchange heat by conduction. The nature of the problem and the results depend strongly on the direction of heat flow. If heat flows into the material with the larger distortivity, the common boundary conditions are sufficient to achieve a solution which satisfies the inequalities associated with a contact problem. For heat flowing in the opposite direction, the common boundary conditions by themselves lead to contradictions, but the difficulties can be overcome by introducing a zone of imperfect contact. The formulation is based on a suitable Green's function, and the problem is reduced to a singular integral equation which must be solved numerically.

Introduction

The common boundary conditions for thermoelastic contact are based on the ideas of perfect contact and perfect insulation. The first idea implies that the interface offers no resistance to heat flow in the regions with solid to solid contact. It is equivalent to an assumption that temperature is continuous across the contact interface. The second idea presumes that no heat is exchanged between the bodies in the separation zones where the solids are out of contact, or that the normal derivative of temperature vanishes in these zones. It is now known, however, that these boundary conditions may lead to mathematical dilemmas for steady-state heat conduction involving contact between bodies with geometrically smooth surfaces. The nature of the difficulties depends on the direction of heat flow: lack of existence for heat flowing into the material with the smaller distortivity (see the list of symbols for definition), and possible lack of uniqueness if heat flows into the material with the larger distortivity.

The difficulty with heat flowing into the material with the smaller distortivity was first noted by Barber [1] in treating the indentation of an elastic half space by a rigid sphere. If the sphere is cold, the contact tractions become tensile near the periphery of the contact region. It was subsequently proven by Barber [2] that the situation cannot be rectified by assuming a concentric array of contact and separation zones. Tensile contact tractions were also encountered by Panek and Dundurs [3] in analyzing the thermoelastic contact between bodies with wavy surfaces. It should be noted that the difficulty is not merely due to an insufficiency of the solution method, but that it is inherent to the problem. Thus Comninou and Dundurs [4] have shown by an asymptotic analysis that a direct transition from perfect contact to separation unavoidably leads to tensile contact tractions, as well as interpenetration of material.

It was conjectured by Dundurs and Comninou [5] on basis of a one-dimensional model that the lack of existence of solutions could be remedied by introducing a pressure-dependent resistance to heat flow in the contact zones. Indeed, it has recently been shown by Duvaut [6] that solutions satisfying the appropriate inequalities (negative normal tractions in the contact zones, and positive gaps in the separation zones) exist for physically realistic laws of the interface resistance. All contact problems are nonlinear because of the inequalities, but a pressure-dependent resistance makes the nonlinearities much stronger.

A modification of the idealized boundary conditions for heat flowing into the material with the smaller distortivity has been proposed by Barber [7]. It pays a penalty in that a new zone (imperfect contact) is needed, but avoids the strong nonlinearities that arise from a pressure-dependent resistance. Accordingly, the contact zone consists of two parts: a zone of perfect contact in which the common assumption of no thermal resistance holds, and a zone of imperfect contact in which the contact pressure vanishes and the contact interface offers some resistance to heat flow. One is led to these boundary conditions by considering a certain limit in the interface resistance, which must be a monotonically decreasing function of pressure [7]. An asymptotic analysis has revealed [4] that the inequalities are not violated at the transition from perfect to imperfect contact if heat flows into the material with the smaller distortivity.

The mathematical difficulties appear to be of the opposite nature when heat flows into the material with the larger distortivity. There

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is evidence that, in such a case, the solutions are not necessarily unique. Thus, if two solids with flat surfaces are pressed together and prevented from bending away from each other globally, one possible steady state of heat conduction corresponds to the solids remaining in contact along the entire interface. It has recently been shown by Comninou and Dundurs [8] that it is also possible to construct solutions involving localized separation zones. A similar conclusion is also implicit in some of the previous results by Dundurs and Panek [9]. Moreover, a particularly simple demonstration of nonuniqueness has been given by Comninou and Dundurs [10] using the Aldo model [11]. Recent work by the authors [12] indicates, however, that it may not generally be possible to achieve uniqueness by introducing a resistance that depends on the pressure in the contact zones and on the gap size in the separation zones.

The situation when one of the contacting bodies has a sharp corner also has been studied [13], but the results are of no immediate interest for the Hertz contact considered, and the subject is mentioned merely for the sake of reference.

The indentation of an elastic half space by a rigid sphere has been treated by Barber for both cases when the sphere is hot and a direct transition from perfect contact to separation is possible [1], and when the sphere is cold and an intermediate zone of imperfect contact is necessary [7]. The same methods could be used to extend this work to two elastic spheres. The present article investigates the planar case of two elastic cylinders. This problem is of interest in its own right, but the main motivation is to provide more detailed results than is feasible in the three-dimensional case.

Mathematical Preliminaries

As it is customary in contact problems of the Hertz type, the geometric profiles of the bodies are approximated in the vicinity of the initial contact as surfaces of second degree, and the boundary conditions are written on their common tangent plane. In other words, the contacting solids are viewed as two elastic half spaces, except that their approximated shapes are incorporated in the boundary conditions to be imposed in the contact region and its immediate vicinity. In two dimensions, the bodies are parabolic cylinders that touch in the undeformed state along the line $x = z = 0$. The initial gap between the bodies, measured along the normal to the tangent plane $y = 0$, is

$$g_0(x) = \tfrac{1}{2}(K_1 + K_2)x^2 \tag{1}$$

where K_1 and K_2 are the curvatures of the cylinders reckoned positive for convexity to the outside. Therefore,

$$dg_0(x)/dx = Kx \tag{2}$$

with

$$K = K_1 + K_2 \tag{3}$$

being the mismatch in curvatures.

The formulation that enforces the required conformity between the bodies in their deformed states is based on a Green's function for interior thermoelastic contact [14]. It consists of a thermoelastic field (heat source and sink) and a purely elastic field (pair of concentrated forces). The advantage of this approach is that most of the boundary conditions pertaining to the problem are satisfied automatically, and that there only remains to find the source-sink and force-pair distributions which enforce a few remaining requirements in the contact zones. The boundary conditions that are automatically embedded in the formulation are continuity of heat flux, continuity of normal tractions, and vanishing shearing tractions at the interface. The full expressions for the field quantities associated with the Green's function are given in reference [14], and we repeat only the relations of immediate interest.

An isolated heat source-sink combination of strength λ acting at the point $(\xi, 0)$ leads to the following quantities at the interface:
Rate of Change of the Temperature Jump:

$$\frac{d\tau(x)}{dx} = \frac{d}{dx}[T_2(x,0) - T_1(x,0)] = \frac{\lambda}{\pi}\frac{k_1 + k_2}{k_1 k_2}\frac{1}{x - \xi} \tag{4}$$

Heat Flux Through the Interface:

$$q(x) = q_y^{(1)}(x,0) = q_y^{(2)}(x,0) = \lambda\delta(x - \xi) \tag{5}$$

Derivative of the Gap:

$$\frac{dg(x)}{dx} = \frac{d}{dx}[u_y^{(1)}(x,0) - u_y^{(2)}(x,0)] = \lambda(\delta_1 - \delta_2)\mathrm{H}(x - \xi) \tag{6}$$

Normal Tractions:

$$N(x) = \sigma_{yy}^{(1)}(x,0) = \sigma_{yy}^{(2)}(x,0) = 0 \tag{7}$$

where $\delta(\ )$ and $\mathrm{H}(\ )$ denote the Dirac and Heaviside functions. If a source-sink combination with the density $\Lambda(x)$ is distributed over a part of the contact interface, the corresponding relations follow from integration with respect to ξ, and

$$\frac{d\tau(x)}{dx} = \frac{1}{\pi}\frac{k_1 + k_2}{k_1 k_2}\int_{-\infty}^{\infty}\frac{\Lambda(\xi)d\xi}{x - \xi} \tag{8}$$

$$q(x) = \Lambda(x) \tag{9}$$

$$\frac{dg(x)}{dx} = (\delta_1 - \delta_2)\int_{-\infty}^{x}\Lambda(\xi)d\xi \tag{10}$$

$$N(x) = 0 \tag{11}$$

A pair of concentrated normal forces of magnitude f_y applied to each of the solids in a tensile direction gives [14]

$$\frac{d\tau(x)}{dx} = q(x) = 0 \tag{12}$$

$$\frac{dg(x)}{dx} = \frac{f_y}{2\pi M}\frac{1}{x - \xi} \tag{13}$$

$$N(x) = f_y\delta(x - \xi) \tag{14}$$

Integration with respect to ξ generates a distribution of interface normal tractions of intensity $F_y(\xi)$, and

$$\frac{d\tau(x)}{dx} = q(x) = 0 \tag{15}$$

Nomenclature

a = half length of perfect contact
b = half length of total contact
$F_y(x)$ = density of a force-pair distribution
f_y = magnitude of a discrete force pair
$g(x)$ = gap between the bodies
$g_0(x)$ = initial gap between the bodies
$\mathrm{H}(\ )$ = Heaviside step function
K = mismatch in curvatures
K_1, K_2 = curvatures of the bodies
k = conductivity
$M = 2\mu_1\mu_2/[\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)]$
$N(x)$ = normal tractions
O = order symbol
P = force transmitted between the bodies (per unit thickness)
Q = rate of heat flow between the bodies (per unit thickness)
$q(x)$ = heat flux through the interface
q_y = component of heat flux
T = temperature
u_y = component of displacement
x, y = coordinates
$\delta = \alpha(1 + \nu)/k$ = distortivity
$\delta(\ )$ = Dirac delta function
$\kappa = 3 - 4\nu$
$\Lambda(x)$ = density of heat source-sink distribution
λ = strength of a discrete source-sink
ν = Poisson's ratio
ξ = integration variable
τ = temperature jump across the interface

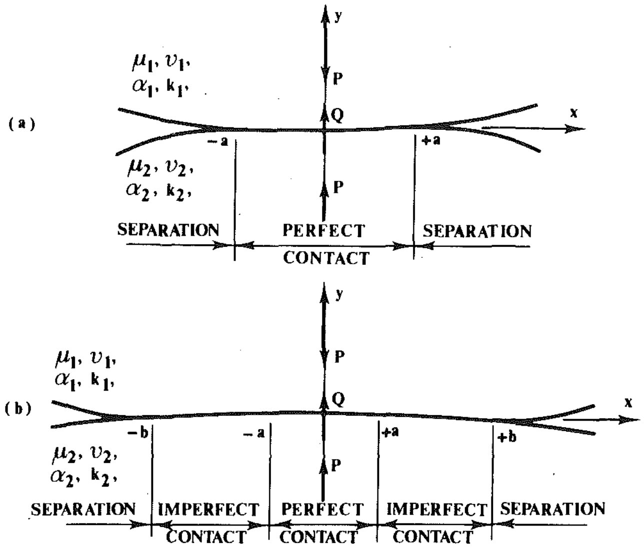

Fig. 1 Geometry of the contacting bodies

$$\frac{dg(x)}{dx} = \frac{1}{2\pi M} \int_{-\infty}^{\infty} \frac{F_y(\xi)d\xi}{x - \xi} \tag{16}$$

$$N(x) = F_y(x) \tag{17}$$

Heat Flowing Into the Material With the Larger Distortivity

If heat flows into the material with the larger distortivity ($\delta_1 > \delta_2$), the central zone of perfect contact can be bordered directly by zones of separation. The corresponding situation is indicated in Fig. 1(a), where the zone of perfect contact extends over the interval $(-a, a)$. The total force pressing the bodies together is denoted by P and the rate at which heat flows through the contact interval by Q ($Q > 0$). Both quantities are reckoned per unit thickness normal to the representative lamina.

As mentioned before, the boundary conditions of continuous heat flux, continuous normal tractions, and vanishing shearing tractions on $-\infty < x < \infty$ are automatically incorporated in the formulation by use of the Green's function [14]. If the source-sink and force-pair distributions are restricted to the interval $-a < x < a$, the separation zones $|x| < a$ are also insulated and free of normal tractions. Consequently, the remaining boundary conditions must only enforce that there be no temperature jump across the interface and the bodies must conform geometrically in the zone of perfect contact, or that

$$\frac{d\tau(x)}{dx} = 0, \quad -a < x < a \tag{18}$$

$$\frac{dg(x)}{dx} = 0, \quad -a < x < a \tag{19}$$

The last condition must only be enforced within an arbitrary constant.

In view of (8), the first boundary condition (18) yields the Cauchy singular integral equation

$$\int_{-a}^{a} \frac{\Lambda(\xi)d\xi}{\xi - x} = 0, \quad -a < x < a \tag{20}$$

with the auxiliary condition

$$\int_{-a}^{a} \Lambda(\xi)d\xi = Q, \quad Q > 0 \tag{21}$$

in which the total heat flux Q is considered as specified. The second boundary condition (19) leads on basis of (2), (10), and (16) to the integral equation

$$\int_{-a}^{a} \frac{F_y(\xi)d\xi}{\xi - x} = 2\pi M \left\{A + Kx + (\delta_1 - \delta_2) \int_{-a}^{x} \Lambda(\xi)d\xi\right\}, \quad -a < x < a \tag{22}$$

where A is an arbitrary constant. The integral equation (22) must be supplemented with the condition

$$\int_{-a}^{a} F_y(\xi)d\xi = -P, \quad P > 0 \tag{23}$$

specifying the total force P transmitted between the bodies. Moreover, the inequalities

$$N(x) < 0, \quad |x| < a \tag{24}$$

$$g(x) > 0, \quad a < |x| \tag{25}$$

must be obeyed by the solution to be constructed.

The solution of the first integral equation (20) together with the auxiliary condition (21) is well known [15]. Thus

$$\Lambda(x) = \frac{Q}{\pi}(a^2 - x^2)^{-1/2} \tag{26}$$

and the heat flux is square-root singular as predicted by the asymptotic analysis of the transition from perfect contact to separation [4].

Substituting (26) into (22)

$$\int_{-a}^{a} \frac{F_y(\xi)d\xi}{\xi - x} = 2\pi M \left\{A + Kx + \frac{Q}{\pi}(\delta_1 - \delta_2)\left(\sin^{-1}\frac{x}{a} + \frac{\pi}{2}\right)\right\} = f(x), \quad -a < x < a \tag{27}$$

Since $F_y(x)$ must be bounded, the consistency condition [15]

$$\int_{-a}^{a} \frac{f(\xi)d\xi}{(a^2 - \xi^2)^{1/2}} = 0 \tag{28}$$

must be satisfied. This yields

$$A + \tfrac{1}{2}(\delta_1 - \delta_2)Q = 0 \tag{29}$$

The solution of (27) is

$$F_y(x) = -\frac{2}{\pi}M(a^2 - x^2)^{1/2}\left\{Kx + \frac{Q}{\pi}(\delta_1 - \delta_2) \times \int_{-a}^{a} \frac{\sin^{-1}(\xi/a)d\xi}{(a^2 - \xi^2)^{1/2}(\xi - x)}\right\}, \quad -a < x < a \tag{30}$$

Applying the auxiliary condition (23) on (30), we obtain after some elementary integrations

$$\frac{P}{MKa^2} = \pi + \frac{4Q(\delta_1 - \delta_2)}{Ka} \tag{31}$$

The singular integral in (30) can be evaluated by the Lobatto-Chebyshev quadrature, as extended to Cauchy integrals by Theocaris and Ioakimidis [16]. Equation (30) is first put in a dimensionless form by the change of variables

$$\xi = ar, \quad x = as$$

The aforementioned quadrature then yields

$$\frac{F_y(s_i)}{MKa} = -\frac{2}{\pi}(1 - s_i^2)^{1/2}\left\{\pi + \frac{Q(\delta_1 - \delta_2)}{Ka}\frac{1}{n-1}\sum_{k=1}^{n}\frac{\lambda_k \sin^{-1} r_k}{r_k - s_i}\right\} \tag{32}$$

where

$$\lambda_k = \begin{cases} \frac{1}{2}, & k = 1, n \\ 1, & k = 2, \ldots, n-1 \end{cases} \tag{33}$$

$$s_i = \cos\frac{(2i-1)\pi}{2(n-1)}, \quad i = 1, \ldots, n-1 \tag{34}$$

$$r_k = \cos\frac{(k-1)\pi}{n-1}, \quad k = 1, \ldots, n \tag{35}$$

The results obtained on this basis are discussed in a later section.

Heat Flowing Into the Material With the Smaller Distortivity

If heat flows into the material with the smaller distortivity ($\delta_1 < \delta_2$), the central zone of perfect contact must be bordered by zones of imperfect contact [4, 7] as indicated in Fig. 1(b). The zone of perfect contact is the interval $|x| < a$, the zones of imperfect contact occupy the intervals $a < |x| < b$. The boundary conditions that must be imposed beyond those satisfied automatically because of the Green's function approach are the same as for heat flow into the material with the larger distortivity, except that they apply to different intervals. Thus

$$\frac{d\tau(x)}{dx} = 0, \quad -a < x < a \tag{36}$$

$$\frac{dg(x)}{dx} = 0, \quad -b < x < b \tag{37}$$

As before, the boundary conditions must be supplemented with auxiliary conditions pertaining to the total rate of heat flow between the bodies and to the force pressing the bodies together. It should also be noted that now the source-sink distribution extends over the interval $-b < x < b$, while the force-pair distribution is restricted to the interval $-a < x < a$.

The boundary condition (36) yields

$$\int_{-b}^{b} \frac{\Lambda(\xi)d\xi}{\xi - x} = 0, \quad -a < x < a \tag{38}$$

and the associated auxiliary condition is

$$\int_{-b}^{b} \Lambda(\xi)d\xi = Q, \quad Q > 0 \tag{39}$$

The other boundary condition (37) gives

$$\int_{-a}^{a} \frac{F_y(\xi)d\xi}{\xi - x} = 2\pi M\left\{A + Kx + (\delta_1 - \delta_2) \times \int_{-b}^{x} \Lambda(\xi)d\xi\right\}, \quad -b < x < b \tag{40}$$

while

$$\int_{-a}^{a} F_y(\xi)d\xi = -P, \quad P > 0 \tag{41}$$

The solution must also satisfy the inequalities [4, 7]

$$N(x) < 0, \quad |x| < a \tag{42}$$

$$\Lambda(x)\tau(x) > 0, \quad a < |x| < b \tag{43}$$

$$g(x) > 0, \quad b < |x| \tag{44}$$

The essential task is to put the system of integral relations (38)–(41) into a form that is suitable for numerical evaluation. Consider first the Cauchy integral in (40). Integrating by parts

$$\int_{-a}^{a} \frac{F_y(\xi)d\xi}{\xi - x} = F_y(\xi)\log|\xi - x|\Big|_{\xi=-a}^{\xi=a} - \int_{-a}^{a} F_y'(\xi)\log|\xi - x|d\xi \tag{45}$$

It is known from the asymptotic analysis of the transition from perfect to imperfect contact that [4]

$$F_y(\xi) = O(a - |\xi|), \quad |\xi| \to a- \tag{46}$$

and consequently

$$\int_{-a}^{a} \frac{F_y(\xi)d\xi}{\xi - x} = -\int_{-a}^{a} F_y'(\xi)\log|\xi - x|d\xi \tag{47}$$

Substituting (47) into (40) and differentiating the resulting expression with respect to x, yields the integral equation

$$\int_{-a}^{a} \frac{F_y'(\xi)d\xi}{\xi - x} = 2\pi M\{K + (\delta_1 - \delta_2)\Lambda(x)\}, \quad -b < x < b \tag{48}$$

which then replaces (40).

From (48)

$$\Lambda(x) = \frac{1}{\delta_1 - \delta_2}\left\{\frac{1}{2\pi M}\int_{-a}^{a} \frac{F_y'(\xi)d\xi}{\xi - x} - K\right\}, \quad -b < x < b \tag{49}$$

and putting (49) into (38)

$$\frac{1}{2\pi M}\int_{-b}^{b} \frac{1}{\xi - x}\int_{-a}^{a} \frac{F_y'(\eta)}{\eta - \xi} d\eta d\xi - K \times \int_{-b}^{b} \frac{d\xi}{\xi - x} = 0, \quad -a < x < a \tag{50}$$

Using the Poincaré-Bertrand formula [15], the double integral in (50) becomes

$$\begin{aligned}\int_{-b}^{b} \frac{1}{\xi - x}\int_{-a}^{a} \frac{F_y'(\eta)}{\eta - \xi} d\eta d\xi &= \int_{-a}^{a} F_y'(\eta)\int_{-b}^{b} \frac{d\xi}{(\xi - x)(\eta - \xi)} d\eta - \pi^2 F_y'(x) \\ &= \int_{-a}^{a} \frac{F_y'(\xi)}{\xi - x}\log\left|\frac{(b - x)(b + \xi)}{(b + x)(b - \xi)}\right| d\xi - \pi^2 F_y'(x)\end{aligned} \tag{51}$$

Thus (50) reduces to

$$\int_{-a}^{a} \frac{F_y'(\xi)}{\xi - x}\log\left|\frac{(b - x)(b + \xi)}{(b + x)(b - \xi)}\right| d\xi - \pi^2 F_y'(x) = 2\pi MK\log\left|\frac{b - x}{b + x}\right|, \quad -a < x < a \tag{52}$$

which is a Fredholm integral equation of the second kind.

Once $F_y'(x)$ is determined, Λ is obtained from (49), and on basis of (8),

$$\frac{d\tau(x)}{dx} = -\frac{1}{\pi}\frac{k_1 + k_2}{k_1 k_2}\int_{-b}^{b} \frac{\Lambda(\xi)d\xi}{\xi - x}, \quad a < |x| \tag{53}$$

We need $d\tau(x)/dx$ in the interval $a < |x| < b$ to check the inequality. In terms of $F_y'(x)$

$$\frac{d\tau(x)}{dx} = \frac{1}{\pi}\frac{k_1 + k_2}{k_1 k_2}\frac{1}{\delta_1 - \delta_2}\left\{K\log\left|\frac{b - x}{b + x}\right| + \frac{1}{2\pi M}\int_{-a}^{a} \frac{F_y'(\xi)}{\xi - x}\log\left|\frac{(b - \xi)(b + x)}{(b + \xi)(b - x)}\right| d\xi\right\}, \quad a < |x| < b \tag{54}$$

and

$$\begin{aligned}\tau(x) = \frac{1}{\pi}\frac{k_1 + k_2}{k_1 k_2}\frac{1}{\delta_1 - \delta_2}&\left\{K\left[(b + x)\log\left(1 + \frac{x}{b}\right) - (b - x)\log\left(1 - \frac{x}{b}\right) + (b - a)\log\left(1 - \frac{a}{b}\right) + (b + a)\log\left(1 + \frac{a}{b}\right)\right]\right. \\ &\left. + \frac{1}{2\pi M}\int_{a}^{x}\int_{-a}^{a} \frac{F_y'(\xi)}{\eta - \xi}\log\left|\frac{(b + \xi)(b - \eta)}{(b - \xi)(b + \eta)}\right| d\xi d\eta\right\}, \\ & a < x < b\end{aligned} \tag{55}$$

In the interval $-b < x < -a$, $\tau(x)$ follows from symmetry, while $F_y(x)$ is verified (numerically) to be even in x.

Finally

$$P = -\int_{-a}^{a} (a - \xi)F_y'(\xi)d\xi \tag{56}$$

and

$$Q = \frac{1}{\delta_1 - \delta_2}\left\{\frac{1}{2\pi M}\int_{-a}^{a} F_y'(\xi)\log\left|\frac{b + \xi}{b - \xi}\right| d\xi - 2Kb\right\} \tag{57}$$

In order to avoid iterations, we take a and b as given and compute the required values of P and Q.

Observing for the kernel in (52) that

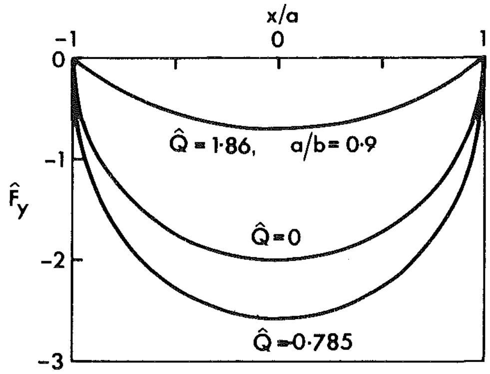

Fig. 2 Comparison of the normalized pressure distributions $\hat{F}_y = F_y/MKa$ for heat flow in different directions with $\hat{Q} = Q(\delta_2 - \delta_1)/Ka$

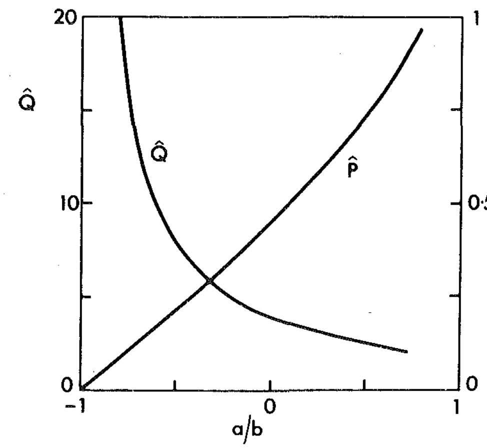

Fig. 3 Relation between a/b, the normalized applied force $\hat{P} = P/MKa^2$ and normalized total heat flow $\hat{Q} = Q(\delta_2 - \delta_1)/Ka$

$$\lim_{x\to\xi}\left\{\frac{1}{\xi - x}\log\frac{(b-x)(b+\xi)}{(b+x)(b-\xi)}\right\} = \frac{1}{b-\xi} + \frac{1}{b+\xi} \tag{58}$$

we use the same collocation (x) and integration points (ξ) to solve (52) numerically. The Lobatto quadrature [17] is convenient for the discretization. The discretized form of (52) according to this quadrature becomes

$$\sum_{j=1}^{n+2} W(r_i, r_j)Y(r_j) = 2\pi \log\left|\frac{1-\lambda r_i}{1+\lambda r_i}\right|, \quad i = 1, \ldots, n+2 \tag{59}$$

where $Y(\xi) = F_y'(\xi)/MK$, $r = \xi/a$, $\lambda = a/b$ and

$$\begin{aligned} W(r_i, r_j) &= \frac{W_j^*}{r_i - r_j}\log\left|\frac{(1-\lambda r_i)(1+\lambda r_j)}{(1+\lambda r_i)(1-\lambda r_j)}\right| \quad &\text{for } i \neq j \\ &= W_j - \pi^2 &\text{for } i = j \end{aligned} \tag{60}$$

$$W_1 = W_{n+2} = \frac{4\lambda}{(1-\lambda^2)(n+1)(n+2)},$$

$$W_1^* = W_{n+2}^* = \frac{2}{(n+1)(n+2)} \tag{61}$$

$$W_{i+1} = \frac{2\lambda A_i}{(1-z_i^2)(1-\lambda^2 z_i^2)}, \quad W_{i+1}^* = \frac{A_i}{1-z_i^2}, \quad i = 1, \ldots, n \tag{62}$$

Furthermore, z_i are the roots of the Jacobi polynomial $P_n^{(1,1)}(z_i) = 0$, A_i the corresponding coefficients and

$$r_{i+1} = z_i, (i = 1, \ldots, n), \quad r_1 = 1, \quad r_{n+2} = -1 \tag{63}$$

The roots z_i and the coefficients A_i are readily obtained by the Fortran program given in reference [18]. It may be noted that the Lobatto quadrature has the advantage of including the end points of the interval in the collocation points. The numerical calculations were performed with double precision, and $n = 40$ was used to obtain enough points for a graphical representation of the results.

The numerical evaluation of $\Lambda(x)$ requires the computation of an integral which is of the Cauchy type (singular) in the interval $|x| < a$. Although the Lobatto quadrature can still be applied in this interval as shown by Theocaris [19], we can no longer choose coinciding collocation and integration points. For best accuracy, the collocation points must be chosed as the roots of appropriate Legendre (or Jacobi) functions of the second kind. Since n is large in our case, the numerical scheme converges also well if we choose for collocation points the midpoints between the integration points. This avoids the need to calculate the zeroes of the aforementioned functions for which no program is available. Convergence was checked in the calculations by doubling n, and no difference was observed in the first eight digits that were printed out. The density $\Lambda(x)$ of the source-sink distribution has logarithmic singularities at $x = \pm a$. Extracting these singularities analytically before discretization did not affect the numerical results significantly.

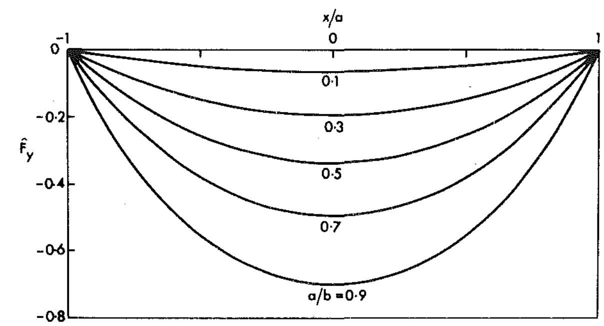

Fig. 4 Normalized pressure distributions $\hat{F}_y = F_y/MKa$ for heat flowing into the material with the smaller distortivity

The Lobbatto quadrature was also used for the evaluation of P, Q and the inner integral in $\tau(x)$. The ordinary trapezoidal rule was used to calculate $F_y(x)$ from $F_y'(x)$. The inequalities and symmetries were also verified numerically.

Results

Typical pressure distributions for heat flowing in either direction are shown in Fig. 2 where $\hat{Q}$ denotes $Q(\delta_2 - \delta_1)/Ka$ and $\hat{P} = P/MKa^2$. The pressure distribution for no heat flow is also included for comparison. It is seen from this figure that, in order to achieve the same extent a of perfect contact, a larger force P must be applied when heat flows into the material with the larger distortivity. It should be noted that the contact pressure distribution has a vertical slope for heat flowing into the material with the larger distortivity, but not for heat flow in the opposite direction. This is in conformity with the results from the asymptotic analysis [4].

Additional results for heat flowing into the material with the smaller distortivity are shown in Figs. 3–6. The relation between a/b, the applied force and total heat flow is shown in Fig. 3. The distribution of the contact pressure is shown in Fig. 4 for different values of a/b. The distribution of heat flux through the interface and the temperature discontinuity in the zone of imperfect contact are given in Figs. 5 and 6 for $a/b = 0.5$. In these figures $\hat{\Lambda} = \Lambda(\delta_2 - \delta_1)/K$ and $\hat{\tau} = \pi k_1 k_2(\delta_2 - \delta_1)\tau/(k_1 + k_2)Ka$. It should be recalled from the asymptotic analysis [4] that the heat flux has a logarithmic singularity at the transition from perfect to imperfect contact.

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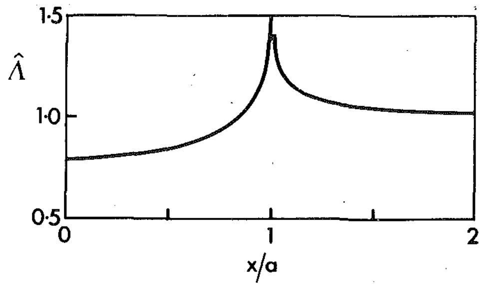

Fig. 5 Distribution of normalized heat flux $\hat{\Lambda} = \Lambda(\delta_2 - \delta_1)/K$ through the interface for $a/b = 0.5$

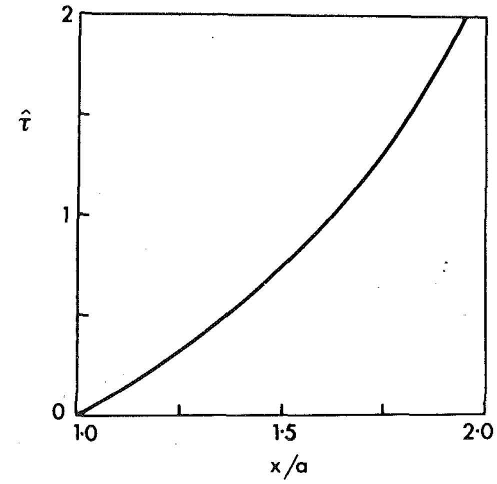

Fig. 6 Normalized temperature discontinuity $\hat{\tau} = \pi k_1 k_2(\delta_2 - \delta_1)\tau/(k_1 + k_2)Ka$ in the zone of imperfect contact for $a/b = 0.5$

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