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Heat Conduction Through a Flat Punch

An elastic half plane is indented by a perfectly conducting rigid flat punch, which is maintained at a different temperature from the half plane. It is found that, depending on the magnitude and direction of the total heat flux, one of the following states occurs: separation at the punch corners, perfect thermal contact throughout the punch face, or an imperfect contact region at the center with adjacent perfect contact regions.

Introduction

This paper is concerned with the steady-state, two-dimensional thermoelastic problem of a frictionless elastic half plane indented by a perfectly conducting rigid flat punch. The punch is maintained at a different temperature from the half plane. In problems of this type, difficulties concerning existence can arise if the punch is cooled and the boundary conditions require continuity of temperature in regions of mechanical contact (perfect thermal contact) and complete insulation in regions of separation. These difficulties have already been exposed [1-4] and will not be elaborated here. They can be avoided by assuming a thermal contact resistance with magnitude inversely related to pressure [3]. The limit, in which the transition from perfect contact to insulation occurs over an infinitesimal range of pressure, leads to the boundary condition of "imperfect contact," introduced in [3], and subsequently used in the solution of a number of thermoelastic contact problems [5, 6].

Statement of the Problem

The geometry of the problem is illustrated in Fig. 1. The rigid punch of width $2b$ is pressed into the half plane $y < 0$ by a compressive force P as shown. The symbols μ , ν , α , k , respectively, denote the shear modulus, Poisson's ratio, coefficient of thermal expansion, and thermal conductivity for the half plane. It is also convenient to define the material constants

Fig. 1 Geometry of the flat punch

$$\kappa = 3 - 4\nu, \quad \delta = \alpha(1 + \nu)/k \quad (1)$$

δ being the thermal distortivity of the half plane.

The total heat flux from the half plane to the punch is Q . The case of most physical interest is $Q > 0$, since it leads to imperfect contact, but results are also given for $Q < 0$ in the interests of completeness.

The problem is analogous to the axisymmetric problem of the cylindrical flat punch treated in [6], although the mathematical formulation is very different. Pursuing this analogy, we anticipate that a central region of imperfect contact will be developed for $Q > 0$ and that separation will occur near the edges for $Q < 0$. However, if P/Q is sufficiently large, perfect contact will be maintained throughout

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the punch face for either direction of heat flow. It is therefore convenient to treat this straightforward case first.

Perfect Contact Solution

If there is perfect thermal contact throughout the punch face, the temperature difference τ and the normal displacement discontinuity, or gap g must be constant in $|x| < b, y = 0$. Using the results and notation of [7], these conditions give

$$\frac{d\tau}{dx} = 0 \quad \text{or} \quad \int_{-b}^b \frac{\Lambda(\xi)d\xi}{\xi - x} = 0, \quad |x| < b \quad (2)$$

$$\frac{dg}{dx} = 0 \quad \text{or} \quad \frac{\kappa + 1}{4\pi\mu} \int_{-b}^b F_y(\xi)d\xi + \delta \int_0^x \Lambda(\xi)d\xi = 0, \quad |x| < b \quad (3)$$

where $\Lambda(\xi)$ is the surface heat flux density out of the half plane and $F_y(\xi)$ is the distribution of normal traction at the interface. The lower limit in the second integral term of (3) is set equal to zero in order to recover the symmetric solution ($F_y(x) = F_y(-x)$) corresponding to the force P being applied along the line of symmetry.

The total heat flux and load are regarded as prescribed quantities and hence we have

$$\int_{-b}^b \Lambda(\xi)d\xi = Q \quad (4)$$

$$\int_{-b}^b F_y(\xi)d\xi = -P \quad (5)$$

The heat-conduction problem defined by (2) and (4) is not influenced by the contact stress distribution and is of standard form with solution

$$\Lambda(x) = \frac{Q}{\pi(b^2 - x^2)^{1/2}} \quad (6)$$

Substituting this result into (3), we find

$$\int_{-b}^b \frac{F_y(\xi)d\xi}{\xi - x} = -\frac{4\mu\delta Q}{\kappa + 1} \sin^{-1}(x/b), \quad |x| < b \quad (7)$$

which has the solution [8]

$$F_y(x) = \frac{4\mu\delta Q}{\pi^2(\kappa + 1)(b^2 - x^2)^{1/2}} \int_{-b}^b \frac{(b^2 - \xi^2)^{1/2} \sin^{-1}(\xi/b)}{\xi - x} d\xi - \frac{C}{(b^2 - x^2)^{1/2}}, \quad |x| < b, \quad (8)$$

Finally, the constant C is determined from the condition (5) for the total load P . Substituting (8) into (4) and performing the integrations we find

$$C = P/\pi \quad (9)$$

and hence

$$F_y(x) = (b^2 - x^2)^{-1/2} \left[-\frac{P}{\pi} + \frac{4\mu\delta Q}{\pi^2(\kappa + 1)} \int_{-b}^b \frac{(b^2 - \xi^2)^{1/2} \sin^{-1}(\xi/b)}{\xi - x} d\xi \right], \quad |x| < b \quad (10)$$

The integral in (10) cannot be evaluated in closed form for general values of x . However, it is an even function of x which varies monotonically from $x = 0$ to $x = b$, the extreme values being

$$\int_{-b}^b \frac{(b^2 - \xi^2)^{1/2} \sin^{-1}(\xi/b)}{\xi} d\xi = 2b(2G - 1) \quad (11)$$

Fig. 2 Geometry for perfect contact and separation

and

$$\int_{-b}^b \frac{(b^2 - \xi^2)^{1/2} \sin^{-1}(\xi/b)}{\xi - b} d\xi = -2b \quad (12)$$

where $G = 0.915965 \dots$ is Catalan's constant [9].

It follows that (10) will define a negative (compressive) contact stress throughout $|x| < b$ if and only if

$$-1.57 < \lambda < 1.89 \quad (13)$$

where

$$\lambda = \frac{4\mu\delta b Q}{(\kappa + 1)P} \quad (14)$$

For $\lambda < -1.57$, corresponding to a sufficiently large heat flux into the half plane from the punch, tensile contact stresses are predicted near $|x| = b$ and we would anticipate separation to occur in this region. For $\lambda > 1.89$, $F_y(x)$ becomes tensile near $x = 0$, but it is shown in [10] that separation cannot occur in such an enclosed region. Also, an asymptotic analysis [4] shows that the smooth transition from perfect thermal contact to separation is impossible for $Q > 0$. We therefore anticipate a region of imperfect contact centered on $x = 0$. These two conditions are treated in the following two sections.

Separation Solution

If $\lambda < -1.57$, separation will occur at the edges of the punch, leaving a region of perfect contact $|x| < c < b$ as shown in Fig. 2. The contact length $2c$ is unknown and must be determined as a function of λ from the continuity condition at the transition. In all other respects the analysis is identical with that of the previous section, c replacing b throughout. In particular,

$$F_y(x) = (c^2 - x^2)^{-1/2} \left[-\frac{P}{\pi} + \frac{4\mu\delta Q}{\pi^2(\kappa + 1)} \int_{-c}^c \frac{(c^2 - \xi^2)^{1/2} \sin^{-1}(\xi/c)}{\xi - x} d\xi \right], \quad |x| < c, \quad (15)$$

(cf. equation (10)).

Using equation (12), it follows that $F_y(x)$ will be bounded at $|x| = c$ if and only if

$$\frac{P}{\pi} + \frac{8\mu\delta c Q}{\pi^2(\kappa + 1)} = 0 \quad (16)$$

or

$$\frac{c}{b} = -\frac{\pi}{2\lambda} \quad (17)$$

Substituting this result into (15) we find

Fig. 3 Geometry for perfect and imperfect contact

$$F_y(x) = -\frac{4\mu\delta Q}{\pi^2} (c^2 - x^2)^{1/2} \int_{-c}^c \frac{\sin^{-1}(\xi/c)}{(c^2 - \xi^2)^{1/2}(\xi - x)} d\xi, \quad |x| < c \quad (18)$$

which is compressive for all $|x| < c$ as required as long as $Q < 0$.

The geometry of the problem demands that $c < b$ and hence $x \leq -1.57$. The separation solution therefore continues the perfect contact solution into large negative values of λ as expected (cf. (13)).

Imperfect Contact Solution

If $\lambda > 1.89$, we anticipate a region of imperfect contact $|x| < a$ enclosed by two strips of perfect contact $a < |x| < b$ as shown in Fig. 3. The boundary condition (3) for mechanical contact applies throughout the punch face and is conveniently written

$$\Lambda(x) = -\frac{(\kappa + 1)}{4\pi\mu\delta} \frac{d}{dx} \int_{-b}^b \frac{F_y(\xi)d\xi}{\xi - x}, \quad |x| < b \quad (19)$$

but equation (2), prescribing continuity of temperature, applies only in $a < |x| < b$. We can express this condition in the integrated form

$$\int_0^b \Lambda(\xi) \log \left| \frac{x^2 - \xi^2}{b^2} \right| d\xi = -C \quad a < |x| < b \quad (20)$$

where C is an arbitrary constant, and the fact that $\Lambda(x)$ is an even function has been used to reduce the range of integration.

In the remainder of the contact area, $|x| < a$, we have imperfect contact and hence the contact stress is infinitesimal [3], i.e.,

$$F_y(x) = 0, \quad |x| < a \quad (21)$$

The contact stress distribution will have a square root singularity at $x = b^-$ and it is convenient to extract it by writing

$$F_y(x) = \frac{A}{(b^2 - x^2)^{1/2}} + G(x), \quad |x| < b \quad (22)$$

where A is a constant. It follows that

$$G(x) = -\frac{A}{(b^2 - x^2)^{1/2}}, \quad |x| < a \quad (23)$$

from (21).

Substituting (22) into (19) we find

$$\Lambda(x) = -\frac{(\kappa + 1)}{4\pi\mu\delta} \frac{d}{dx} \left[\int_{-b}^b \frac{A}{(b^2 - \xi^2)^{1/2}} \frac{1}{\xi - x} d\xi + \int_{-b}^b \frac{G(\xi)}{\xi - x} d\xi \right], \quad (24)$$

but the first term vanishes leaving

$$\Lambda(x) = -\frac{\kappa + 1}{4\pi\mu\delta} \frac{d}{dx} \int_{-b}^b \frac{G(\xi)}{\xi - x} d\xi, \quad |x| < b \quad (25)$$

Finally, we substitute this expression into (20), noting that $G(x)$ is an even function and using (23) to evaluate the integral in $|x| < a$, obtaining

$$\int_0^b \log \left| \frac{x^2 - \xi^2}{b^2} \right| \frac{d}{d\xi} \left[\int_a^b G(\eta) \left(\frac{1}{\eta - \xi} - \frac{1}{\eta + \xi} \right) d\eta \right] d\xi - A \int_0^a \left(\frac{1}{\eta - \xi} - \frac{1}{\eta + \xi} \right) \frac{d\eta}{(b^2 - \eta^2)^{1/2}} = -C, \quad a < x < b \quad (26)$$

Integrating by parts and using the Poincaré-Bertrand result to reverse the order of integration we have

$$\pi^2 G(x) - \int_a^b G(\eta) \left[K(\eta, x) + \frac{2b}{b^2 - \eta^2} \log \left(1 - \frac{x^2}{b^2} \right) \right] d\eta + \frac{2aA}{b(b^2 - a^2)^{1/2}} \log \left(1 - \frac{x^2}{b^2} \right) + A \int_0^a \frac{K(\eta, x)}{(b^2 - \eta^2)^{1/2}} d\eta = -C, \quad a < x < b, \quad (27)$$

where

$$K(\eta, x) = \frac{1}{\eta - x} \log \left| \frac{(b - x)(b + \eta)}{(b + x)(b - \eta)} \right| + \frac{1}{\eta + x} \log \left| \frac{(b + x)(b + \eta)}{(b - x)(b - \eta)} \right| \quad (28)$$

This integral equation contains two unknown constants, A , C , and the unknown extent of the contact region a . It is convenient to regard a as a prescribed quantity and to allow Q (equation (51)) to float. We then have equation (4) for P (essentially a scaling condition) and a continuity condition at $x = a^+$ to determine the remaining constants.

Asymptotic arguments [4] show that $F_y(x)$ must tend to zero at the edge of the perfect contact region and hence

$$G(a^+) = -\frac{A}{(b^2 - a^2)^{1/2}} \quad (29)$$

This equation is used to substitute for A in (27), which can now be solved numerically (see next section) for the function $G(x)/C$. We then find A/C from (29) and C from (4), which can be put in the form

$$P = -2A \cos^{-1}(a/b) - 2 \int_a^b G(\xi) d\xi \quad (30)$$

using (22).

The total heat flux—now a dependent variable for given a/b —can be obtained by substituting (25) into (5) and is

$$Q = \frac{\kappa + 1}{2\pi\mu\delta} \int_a^b \frac{2bG(\eta)}{b^2 - \eta^2} d\eta - \frac{2aA}{b(b^2 - a^2)^{1/2}} \quad (31)$$

while the distribution of heat flux through the contact area is

$$\Lambda(x) = -\frac{\kappa + 1}{4\pi\mu\delta} \frac{d}{dx} \left\{ \int_a^b \frac{2\xi G(\xi) d\xi}{\xi^2 - x^2} + \frac{A}{(b^2 - x^2)^{1/2}} \log \left| \frac{(a - x)[b^2 + ax - (b^2 - a^2)^{1/2}(b^2 - x^2)^{1/2}]}{(a + x)[b^2 - ax - (b^2 - a^2)^{1/2}(b^2 - x^2)^{1/2}]} \right| \right\} \quad a < x < b, \quad (32)$$

from (25) and (23). It may be verified that $\Lambda(x)$ exhibits correctly the singular behavior predicted by the asymptotic analysis [4].

Numerical Solution

The Cauchy integrals appearing in the expression for $F_y(x)$ (equations (10) and (18)) were evaluated using the Erdogan and Gupta quadrature [11]. For the numerical integration of the Fredholm integral equation (27) the Gauss-Jacobi quadrature [12] with weight function 1 was used. Accordingly, the integration points were chosen as the roots of Legendre polynomials. The collocation points were also chosen as the roots of the same polynomials and L'Hôpital's rule was applied when necessary.

Fig. 4(a) Contact pressure distribution for perfect contact and separation ($NPR = F_y(x)b/P$, $X/B = x/b$)

Fig. 4(b) Contact pressure distribution for perfect contact ($NPR = F_y(x)b/P$, $X/B = x/b$)

Fig. 4(c) Contact pressure distribution for perfect and imperfect contact

Results and Discussion

The state of the system is best described in terms of the nondimensional parameter λ defined by equation (14).

For $\lambda < -1.57$, separation occurs at the punch corners, leaving a contact width $2c$ which varies inversely with λ as defined by equation (17). Typical contact pressure distributions for this region are shown nondimensionally in Fig. 4(a) for $c/b = 0.25, 0.5, 0.75, 1$. Note that the contact pressure tends to zero at the edge of the contact region as in the isothermal Hertzian problem.

Fig. 5 Ratio a/b versus λ

Fig. 6 Typical heat flux distribution

For $-1.57 < \lambda < 1.89$, perfect contact is maintained throughout the punch face and there is a square root singularity in pressure at the sharp edge of the punch except for the transitional case $\lambda = -\pi/2$. Typical pressure profiles are shown in Fig. 4(b). Notice that as λ is increased the pressure becomes more concentrated at the edges of the punch at the expense of the center until, at $\lambda = 1.89$, the central pressure is zero.

For $\lambda > 1.89$, a central region of imperfect contact of width $2a$ is developed. The ratio a/b increases monotonically with λ in this range as shown in Fig. 5. Corresponding pressure distributions are given in Fig. 4(c) and exhibit a square root singularity near $x = b$ and a linear behavior near $x = a$ as required by the asymptotic analysis.

A typical heat flux distribution (for the case $a/b = 0.5$) is shown in Fig. 6. There is always a square root singularity in flux at the outer edge of the contact region, but a weaker (logarithmic) singularity is developed on both sides of the perfect-imperfect contact transition.

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