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Stability Considerations in Thermoelastic Contact

A simple one-dimensional model is described in which thermoelastic contact conditions give rise to nonuniqueness of solution. The stability of the various steady-state solutions discovered is investigated using a perturbation method. The results can be expressed in terms of the minimization of a certain energy function, but the authors have so far been unable to justify the use of such a function from first principles in view of the nonconservative nature of the system.

Introduction

It is known that mathematical difficulties arise in problems of thermoelastic contact between bodies with geometrically smooth surfaces. In particular the conventional boundary conditions of perfect thermal contact (no resistance to heat flow leading to continuity of temperature) in regions of mechanical contact and complete insulation (no heat flux) in regions of separation lead to ill-posed boundary-value problems whenever the hotter solid has the higher thermal distortivity defined by

$$\delta = \alpha(1 + \nu)/k \tag{1}$$

where α, k, ν are, respectively, the coefficient of linear thermal expansion, thermal conductivity, and Poisson's ratio for the material [1–3]. An asymptotic analysis of the transitions between the various boundary regions [3] suggests that the conventional boundary conditions can be safely applied when heat flows in the opposite direction, but there is evidence that, in this case, the solution obtained is not necessarily unique [4]. This immediately raises the question of stability which forms the subject of this paper.

Stability questions can be probed by energy arguments or by an analysis of small perturbations about the steady state. The former would have the advantage of analytic simplicity, since a perturbation analysis involves a consideration of transient heat conduction, but it is far from clear what energy formulation would be appropriate for thermoelastic contact, since the system is inherently nonconservative, i.e., it is possible to devise loading cycles which would cause the contacting solids to act as a heat engine. In order to elucidate this question and to investigate the fundamental characteristics of thermoelastic contact, we give here an exhaustive treatment of the simplest contact system which exhibits thermoelastic nonuniqueness—a one-dimensional rod conducting heat between rigid walls. A perturbation analysis is used to determine the conditions for stability of the various steady-state solutions, but it transpires that these conditions can be stated in terms of the minimization of an energy function with a straightforward physical interpretation.

The Model: Steady-State Solutions

The system to be analyzed is illustrated in Fig. 1. Two perfectly conducting and rigid walls, separated by a distance l, are maintained at temperatures T_0 and zero, respectively, and a uniform elastic rod of cross-sectional area A is built into the cold wall as shown. The length of the rod is such as to leave a gap $g = g_0$ when the temperature is everywhere zero. The same model, with the temperature difference reversed, has been used to investigate the nonexistence of solutions in thermoelastic contact [2].

For sufficiently high values of the hot wall temperature T_0, we should expect two steady-state solutions to the problem: One involving contact between the rod and the wall, the gap being closed by thermal expansion of the rod; the other with separation between the rod and the wall, the gap being sufficient to prevent significant heat flow into the rod. To investigate the matter further, we postulate the existence of a thermal resistance R between the hot wall and the rod, which will be a function of contact force P when contact occurs and of gap size g when it does not. No assumptions will be made about the nature of this function though, on physical grounds, we should expect

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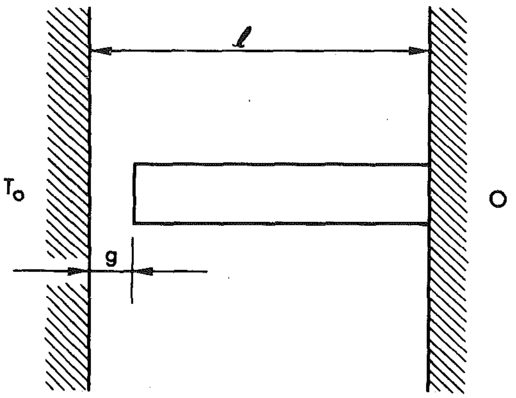

Fig. 1 The rod transferring heat from the hot to the cold wall

it to fall monotonically as the gap is reduced or the pressure increased.

If the contact resistance R is known, the temperature T' at the hot end of the rod can be determined from continuity of heat flux Q in the steady state. Thus

$$Q = (T_0 - T')/R = T'Ak/l \tag{2}$$

Solving for T', we have

$$T' = \frac{T_0}{1 + AkR/l} \tag{3}$$

and hence the unrestrained thermal expansion of the rod is

$$u_{th} = \frac{1}{2}\alpha T' l = \frac{\alpha l^2 T_0}{2(l + AkR)} = u_0 f \tag{4}$$

where $u_0 = \frac{1}{2} \alpha l T_0$ is the thermal expansion which would be developed with $T' = T_0$, i.e., perfect thermal contact between the rod and the wall, and

$$f = \frac{l}{l + AkR} \tag{5}$$

The function f ranges from zero for complete thermal insulation ($g \to \infty$) to unity for perfect thermal contact ($P \to \infty$).

We can now write down the equations determining the steady-state solutions. For separation,

$$g = g_0 - u_0 f(g), \quad g \geq 0 \tag{6a}$$

whereas for contact,

$$0 = g_0 - u_0 f(P) + Pl/AE, \quad P \geq 0 \tag{6b}$$

The variables g and P apply to separate regimes which intersect only in the point $g = P = 0$. We can therefore define a continuation of g into negative values by the relation

$$g = -Pl/AE, \quad P > 0 \quad \text{and} \quad g < 0 \tag{7}$$

With this definition, the two equations (6a, b) reduce to the same form which is conveniently written

$$f(g) = (g_0 - g)/u_0 \tag{8}$$

A graphical solution to equation (8) could be envisaged as shown in Fig. 2. The two sides of the equation are plotted as separate functions of g, and steady-state solutions are represented by intersections between these functions. In general, there will be either one or three solutions, depending on the values of g_0, u_0 and the nature of the function $f(g)$.

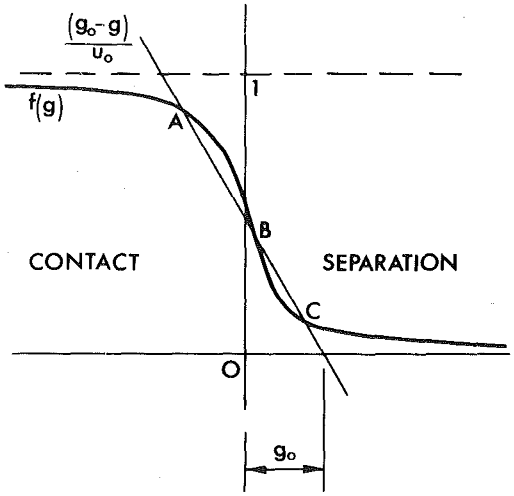

Fig. 2 Graphical interpretation of the stability criterion

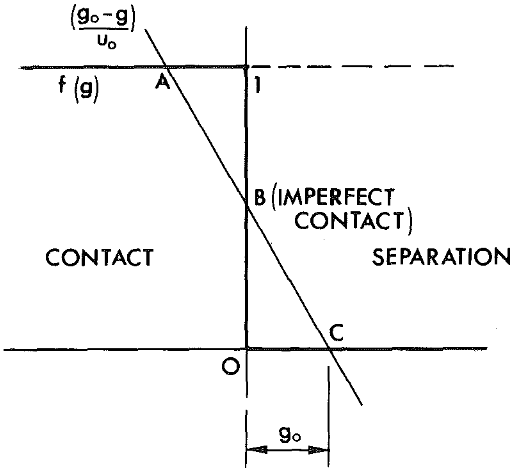

Fig. 3 Graphical interpretation for the idealized boundary conditions

It is instructive to examine the limiting case where the contact resistance passes from zero to infinity over an infinitesimal range of values about $g = 0$. The corresponding limit for $f(g)$ is the step function $f(g) = H(-g)$ shown in Fig. 3. When three steady-state solutions occur, two of them lie on the horizontal branches of the step functions at A and C, corresponding to perfect thermal contact and separation, respectively, while the third lies on the vertical step at B and corresponds to the state defined by a similar limiting process as "imperfect contact" in reference [2].

Stability Analysis

In order to investigate the stability of the various steady states described by equation (8), we examine the conditions under which a small perturbation can grow exponentially with time. Such a perturbation will only be possible for certain eigenvalues of the exponential growth rate, and the condition for stability is that there should be no positive eigenvalues. If complex eigenvalues are possible—corresponding to exponentially growing oscillatory perturbations—there must be none with a positive real part.

Temperature Distribution in the Bar. The perturbation in temperature and heat flux in the bar from the steady-state value must satisfy the transient heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{9}$$

where κ is the thermal diffusivity of the bar material and x is measured along the bar from the cold end.

Assuming a perturbation of the form $T = \phi(x)e^{at}$, we have

$$\frac{d^2\phi}{dx^2} - \frac{a}{\kappa}\phi = 0 \tag{10}$$

the solution of which is

$$\phi = A \cosh \lambda x + B \sinh \lambda x \tag{11}$$

with $\lambda = (a/\kappa)^{1/2}$. Applying the condition $T = 0$ at $x = 0$, the constant A must be zero, and hence the perturbation in temperature is described by

$$T = Be^{at} \sinh \lambda x \tag{12}$$

while the heat flux is

$$Q_x = -Ak\frac{\partial T}{\partial x} = -BAk\lambda e^{at} \cosh \lambda x \tag{13}$$

The change in the gap between the rod and the wall due to this temperature perturbation is

$$\Delta g = -\int_0^l \alpha T dx = -B\frac{\alpha}{\lambda}e^{at}(\cosh \lambda l - 1) \tag{14}$$

and the perturbations in heat flux Q and temperature T at the free end $x = l$ are

$$\Delta Q = -Q_x(l) = BAk\lambda e^{at} \cosh \lambda l \tag{15}$$

$$\Delta T = T(l) = Be^{at} \sinh \lambda l \tag{16}$$

Contact Resistance Equation. To complete the analysis, we linearize the relation (2) between heat flux and temperature in the vicinity of the steady-state solution, obtaining

$$\Delta Q = -\frac{(T_0 - T')\Delta R}{R^2} - \frac{\Delta T}{R} \tag{17}$$

where ΔR is the corresponding perturbation in contact resistance and R, T' here describe the steady-state values.

Equation (17) can be cast in terms of the function $f(g)$ by using equation (5). Thus

$$R = \frac{l}{Ak}\left(\frac{1}{f} - 1\right) \tag{18}$$

and hence

$$\Delta R = \frac{\partial R}{\partial g}\Delta g = -\frac{lf'\Delta g}{Akf^2} \tag{19}$$

Substituting for T', R, ΔR from equations (3), (18), and (19), respectively, into equation (17), we obtain

$$\frac{l}{Ak}\left(\frac{1}{f} - 1\right)\Delta Q = \frac{T_0 f'\Delta g}{f} - \Delta T \tag{20}$$

Characteristic Equation. Finally, we substitute for the perturbations Δg, ΔQ, ΔT from equations (14)–(16) into (20) to obtain the characteristic equation for a which is

$$B\lambda l\left(\frac{1}{f} - 1\right)e^{at} \cosh \lambda l = -B\alpha T_0 \frac{f'}{\lambda f}e^{at}(\cosh \lambda l - 1) - Be^{at} \sinh \lambda l \tag{21}$$

and which can be simplified to the form

$$(1 - f)y^2 \cosh y + 2u_0 f'(\cosh y - 1) + fy \sinh y = 0 \tag{22}$$

where

$$y^2 = al^2/\kappa \tag{23}$$

Restricting attention initially to the case of real roots, we expand equation (22) in the form

$$\sum_{j=1}^{\infty} \frac{y^{2j}}{(2j)!}[2j(2j - 1)(1 - f) + 2jf + 2u_0 f'] = 0 \tag{24}$$

The functions f, $(1 - f)$ are both positive for positive values of R (see the section, "The Model: Steady-State Solutions") and hence the series will be positive for large values of y. However, it will be negative at small values of y, giving a zero somewhere on the real axis if

$$2(1 - f) + 2f + 2u_0 f' < 0 \tag{25}$$

or

$$-f' > 1/u_0 \tag{26}$$

(Note that f' is generally negative—see Fig. 2). Furthermore, it is clear that if this condition is *not* satisfied, all the terms of equation (24) will be positive and there will be no real root for y (except the trivial solution $y = 0$).

Of course, equation (22) may have complex roots, describing oscillatory perturbations. This possibility is investigated in the Appendix, where it is shown that (26) also describes the condition for a complex root for the exponential growth rate a to have a positive real part. Hence, we conclude that the system is unstable *if and only if* condition (26) is satisfied.

Graphical Interpretation and History Dependence. By referring to equation (8) it is clear that the criterion (26) for instability describes those intersections in Fig. 2 at which the function $f(g)$ crosses $(g_0 - g)/u_0$ from above with increasing g. Thus solution B is unstable, while A and C are stable.

The steady-state solution C is possible only if the imposed temperature T_0 is smaller than a certain temperature, say, T_C which can be determined graphically for given g_0 from the curve $f(g)$ in Fig. 2. Similarly, solution A can occur only if T_0 is above a certain temperature, say, T_A. All three solutions A, B, and C are possible in the intermediate range $T_C < T_0 < T_A$. Which of the stable steady states are reached depends on the history of the thermal process. Suppose that the rod has a certain initial temperature distribution and that, during the early stages, T_0 depends on time, but that later in the process T_0 is kept constant so that a steady-state distribution of temperature in the rod is eventually achieved. It is then clear that the final state depends on the previous manipulations of the process. Consider for instance the rod being initially at zero temperature with T_0 slowly raised from zero to some finite value. In such case, the steady state reached will correspond to solution C, provided the final value of T_0 is smaller than T_C. If, on the other hand, the rod has initially a temperature distribution such that it is in contact with the hot wall, and the temperature T_0 is not suddenly dropped below a level to break contact, the steady state A will establish itself for long time values of $T_0 > T_A$. The unstable steady state B could conceivably be reached by carefully steering the process during its early stages. However, any temperature disturbance that corresponds to thermal elongation of the rod will eventually make the system settle down in state A. Conversely, disturbances that make the rod contract slightly will make it go into state C.

More generally, we conclude that, if the contact resistance is a continuous function of g, there will be an odd number of steady-state solutions which are alternately stable and unstable. The stable solutions might be thought of as separated from each other by "higher energy" unstable barriers. In the particular case of the step change in resistance shown in Fig. 3, the imperfect contact solution acts as such a barrier and is unstable.

Energy Considerations

If we define an "energy function"

$$U(g) = \frac{AE}{l}\left[\frac{1}{2}(g_0 - g)^2 + \int u_0 f dg\right] \tag{27}$$

with E denoting Young's modulus, the condition (8) for a steady-state solution can be expressed as

$$\partial U/\partial g = 0 \tag{28}$$

while the condition for instability (26) is

$$\partial^2 U/\partial g^2 < 0 \tag{29}$$

In other words, the function U is stationary at all steady-state solutions, being a maximum if the solution is unstable and a minimum if it is stable. Thus it behaves in all respects as the total energy of a conservative mechanical system.

Furthermore, we can give a physical interpretation to the two terms in equation (27). The first term, $\frac{1}{2}(g_0 - g)^2 AE/l$, is the elastic strain energy involved in extending or compressing the rod isothermally at temperature zero, while the second term can be expressed as

$$-\int u_0 f dP = -\int u_{th} dP \qquad (30)$$

(see equations (4) and (7)) where the compressive force P has a continuation $(-gAE/l)$ into negative values.

As long as conditions are changed slowly enough for the temperature distribution in the rod to be in a quasi-steady state, the rod with pressure/gap dependent contact resistance will exhibit a unique relation between load (gap) and extension. Now, if a mechanical system could be constructed with the *same* load extension relation, the normal energy theorems could be applied to it, since we should not now have a continuous flow of energy across a boundary containing the system. However, such a mechanical system would only be conservative if the load was always varied incrementally, i.e., the sudden application or removal of a finite load may lead to the system doing extra work on the surroundings. Of course, this quasi-static, incremental behavior cannot be guaranteed in the thermal system, but the energy function obtained in the foregoing from perturbation arguments is closely related to that which would be obtained by imposing the requirement of minimum complementary energy on such a system.

The authors have as yet been unable to justify use of such an energy argument other than through the perturbation analysis set out in the section, "Stability Analysis," but the result summarized in equations (27)–(29) is extremely suggestive. If such a justification could be produced, it would be capable of extension to more difficult thermoelastic contact problems involving nonuniqueness, such as those concerned with the half space [4], for which a perturbation analysis would be of formidable complexity. The reader's attention is drawn to this unsolved problem.

References

1 Barber, J. R., "Indentation of the Semi-infinite Elastic Solid by a Hot Sphere," *International Journal of Mechanical Sciences*, Vol. 15, 1973, pp. 813–819.

2 Barber, J. R., "Contact Problems Involving a Cooled Punch," *Journal of Elasticity*, Vol. 8, 1978, pp. 409–423.

3 Comninou, M., and Dundurs, J., "On the Barber Boundary Conditions for Thermoelastic Contact," ASME JOURNAL OF APPLIED MECHANICS, Vol. 46, 1979, pp. 849–853.

4 Comninou, M., and Dundurs, J. "On Lack of Uniqueness in Heat Conduction Through a Solid to Solid Interface," ASME *Journal of Heat Transfer*, Vol. 102, 1980, pp. 319–323.

APPENDIX

Criterion for Complex Roots of Equation (22)

We are concerned to find the conditions under which the function

$$F(z) = z^2 \cosh z + \alpha z \sinh z + \beta(\cosh z - 1) \qquad (31)$$

has complex roots corresponding to values of z^2 with positive real part. In the z-plane, the corresponding zeros of $F(z)$ must lie in the two sectors shaded in Fig. 4 and bounded by the lines $z = \pm(1 \pm i)\omega$. The origin is also excluded by two small quarter circles, as a zero there is introduced by the multiplication by y in the derivation of equation (22) and is of no physical significance.

$F(z)$ is a continuous function of the coefficients $\alpha = f/(1-f)$ and $\beta = 2u_0 f'/(1-f)$ and has no zeros in the domains of Fig. 4 if $\alpha = \beta = 0$. Hence, if we start from this condition and change α, β continuously, the zeros will move continuously about the complex plane and will only be able to enter the domain by crossing one of its boundaries. The stability boundary is therefore equivalent to the condition that $F(z)$ has a zero on the domain boundary. We consider these boundaries in turn:

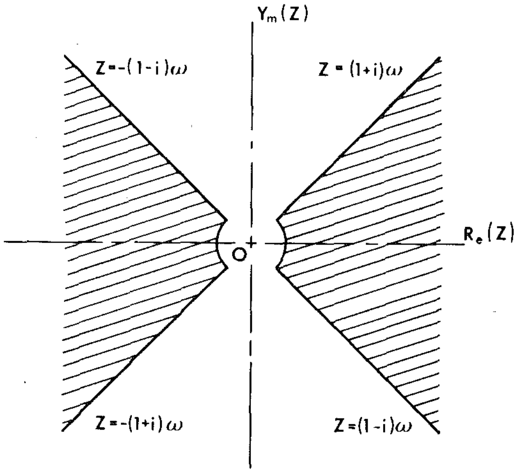

Fig. 4 Regions in the complex plane

(*i*) $z = (1+i)\omega$. If we treat (31) as an equation for α, we have

$$\alpha = -\frac{z^2 \cosh z + \beta(\cosh z - 1)}{z \sinh z} \qquad (32)$$

which can be decomposed in the form

$$\alpha = \frac{2\omega^2 sh \cdot s - \beta(ch \cdot c - 1) - i(2\omega^2 ch \cdot c + \beta sh \cdot s)}{\omega[sh \cdot c - s \cdot ch + i(sh \cdot c + ch \cdot s)]} \qquad (33)$$

where $s = \sin \omega$, $c = \cos \omega$, $sh = \sinh \omega$, $ch = \cosh \omega$.

Now α is a *real* constant and hence the imaginary part of equation (33) is zero, i.e.,

$$[2\omega^2 sh \cdot s - \beta(ch \cdot c - 1)](sh \cdot c + ch \cdot s) - [2\omega^2 ch \cdot c + \beta sh \cdot s](sh \cdot c - ch \cdot s) = 0 \qquad (34)$$

from which, after some manipulation we have

$$\beta = \frac{2\omega^2(ch \cdot sh - c \cdot s)}{(sh - s)(ch - c)} \qquad (35)$$

Back substitution into equation (33) then gives

$$\alpha = -\frac{\omega(ch \cdot sh^2 - c \cdot s^2)}{2(sh^2 \cdot c^2 + ch^2 \cdot s^2)(sh - s)} \qquad (36)$$

which is negative for all ω, whereas $\alpha = f/(1-f)$ must be positive for all possible resistance functions f.

We conclude that no zero can enter the domain across $z = (1+i)\omega$. A similar argument can be used to prove that zeros cannot enter across the other diagonal boundaries.

(*ii*) $z = re^{i\theta}$, *r small*, $-\pi/4 < \theta < \pi/4$. For this boundary, $F(z)$ can be expanded as in equation (24) and condition (25) is immediately obtained on dropping all except the lowest order terms in r.

(*iii*) *The Point at Infinity.* As $z \rightarrow \infty$, the function $F(z)$ becomes dominated by the term $z^2 \cosh z$ which is insensitive to variation of α, β. Thus no zeros can enter the domain through the point at infinity.

We therefore conclude that condition (25) is the correct stability criterion for real or complex roots. In fact, a further analysis along the same lines, but excluding the real axis by the two lines $z = \pm i\delta$, shows that zeros of $F(z)$ in the domain of Fig. 4 can only occur on the real axis. In other words, there are no oscillatory perturbation solutions to the problem.