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Stability of thermoelastic contact

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SYNOPSIS The stability of nominally uniform contact between two elastic half-planes is investigated, assuming a pressure/gap dependent interface resistance.

Long wavelength sinusoidal perturbations in pressure are generally unstable, if the heat flows into the more distortive body, the critical wavelength depending on the contact resistance function. Perturbations grow at different rates and there is an optimum range of wavelengths which will eventually dominate the transient process.

A remarkable conclusion is that in some cases the steady-state solution with uniform pressure is unique, but unstable, indicating that oscillatory behaviour must occur.

1 INTRODUCTION

When elastic contact occurs between two conforming or nearly conforming bodies, relatively small changes in surface profile can have a substantial effect on the contact pressure distribution and on the magnitude of the contact area. We should not be surprised, therefore, to find that thermoelastic deformations - though small in most engineering applications - are often sufficient to have a major effect on the mechanics of contact.

Solutions of classical thermoelastic contact problems - such as the Hertzian contact of two spherical bodies at different temperatures [1,2] - confirm this conclusion and also expose the striking result that neither existence nor uniqueness theorems can be proved for such problems with conventional idealized boundary conditions.

Barber [2] showed that difficulties over existence can be overcome in a one-dimensional system by postulating a continuous pressure dependent thermal contact resistance at the interface and Duvaut [3] has proved an existence theorem for a boundary condition of this form. Duvaut also proved that the thermoelastic contact problem is unique if the contact resistance varies sufficiently gradually with pressure, but experience with specific problems suggests that this is too severe a restriction and that multiple solutions may occur in practical situations. For example, Srinivasan and France [4] report evidence of erratic performance of duplex heat exchanger tubes which they attribute to the existence of multiple steady-state solutions to the contact problem at the interface between the two component tubes, this conclusion being supported by a computer simulation of the contact problem utilizing empirically obtained data for the thermal contact resistance.

The possibility of multiple solutions raises questions of stability, which have been investigated for various one-dimensional systems, using perturbation methods [5,6,7]. In all these cases, it was found that when the steady-state solution is unique it is also stable, whereas when multiple solutions are obtained, they are alternately stable and unstable.

Nothing is yet known about the stability of two and three dimensional thermoelastic contact, though Comninou and Dundurs [8] have shown that two dissimilar half-planes may exist in a steady-state involving periodic contact and separation zones in addition to the trivial state with uniform contact pressure. Also, Richmond and Huang [9] have suggested that the growth of a sinusoidal perturbation in an otherwise uniform contact pressure between a solidifying casting and the mould may be responsible for experimentally observed waviness in nominally plane cast surfaces. It seems probable that the occurrence of solidification is not a prerequisite for such behaviour and that a sinusoidal perturbation between two contacting half-planes may be unstable because of the interaction of thermoelastic distortion and a pressure dependent thermal contact resistance. This is the question to be investigated in the present paper.

2 FORMULATION

We consider the problem of two half-planes, $y>0$ and $y<0$, making contact at the common plane $y = 0$ (see Figure 1). The half-planes are pressed together by a uniform pressure p_0 and transmit a uniform heat flux, $q_y = q_0$ in the positive y direction. The half-planes are of different materials, the appropriate material properties being distinguished by the suffix 1, for the half-plane $y>0$, and 2 for the half-plane $y<0$.

Some readers may feel that the infinite extent of the contact plane lends artificiality to the problem and may introduce spurious effects. An alternative model is that of two thin walled cylinders of radius, r, with plane ends, pressed together by a force, P, and transmitting a heat flux, Q (see Figure 2.). Provided shell bending effects can be neglected, this system can be 'unwrapped' to give a plane stress problem for two contacting half-planes, the only restriction being that the solution must be periodic in x with period $2\pi r$. The effect of shell bending in an isothermal problem for this geometry is discussed by Azarkhin and Barber [10]. We shall return to this problem in the discussion.

We postulate the existence of a thermal contact resistance, R at the interface, which is a function of contact pressure. No restrictions are imposed on the nature of this function except that it be continuous. However, typical experimental results are given in [4] and are shown in Figure 3. We note that the resistance is very sensitive to contact pressure when the pressure is low.

3 THE TEMPERATURE PERTURBATION

The system of Figure 1 (or Figure 2) clearly has a trivial steady-state solution in which the contact pressure is uniform and equal to p_0. In this case, the contact resistance is $R(p_0)$, the temperature is everywhere linear in y and independent of x and the heat flux is also independent of x and equal to q_0.

We investigate the conditions under which a perturbation in this steady-state solution which is sinusoidal in x can grow exponentially in time. The perturbation in temperature is therefore written in the form

$$T = f(y)e^{bt}\cos(mx) \qquad (1)$$

where the function f(y) has to be chosen to satisfy the transient heat conduction equation

$$\nabla^2 T = \frac{1}{k_i}\frac{\partial T}{\partial t} \qquad (2)$$

where k_i (i = 1,2) is the thermal diffusivity of the material.

Substituting (1) into (2) and solving for f(y), we find that the perturbation in temperature in the two half-planes can be written

$$T = A_i \exp(bt - a_i y)\cos(mx) \qquad (3)$$

where A_i are two arbitrary constants and

$$a_i^2 = m^2 + b/k_i \qquad (4)$$

The perturbation must decay away from the contact plane and hence we must take the positive root of equation (4) for a_1 and the negative root for a_2.

4 THERMOELASTIC STRESSES AND DISPLACEMENTS

A particular solution of the thermoelastic problem corresponding to the temperature field of equation (3) can be obtained in terms of a thermoelastic potential, ϕ , where

$$2\mu\underline{u} = \underline{\nabla}\phi \qquad (5)$$

and

$$\nabla^2\phi = \frac{2\mu\alpha(1+\nu)T}{(1-\nu)} \qquad (6)$$

(see Westergaard [11], section 64). In these equations, α is the coefficient of thermal expansion, μ is the modulus of rigidity and ν is Poisson's ratio of the material.

It can be verified by substitution that the potential function

$$\phi_1 = \frac{2\mu_1\alpha_1(1+\nu_1)k_1A_1}{(1-\nu_1)b}\{e^{-a_1y} - \frac{a_1}{m}e^{-my}\} \cdot e^{bt}\cos(mx) \qquad (7)$$

satisfies equations (3,6), for the half-plane y>0. The corresponding tractions and normal displacement at the contact plane y = 0 are

$$\sigma_{xy} = \frac{\partial^2\phi_1}{\partial x\partial y} = 0 \quad ; \quad y = 0 \qquad (8)$$

$$\sigma_{yy} = -\frac{\partial^2\phi_1}{\partial x^2} = -\frac{2\mu_1\alpha_1(1+\nu_1)A_1 m}{(1-\nu_1)(a_1+m)} \cdot e^{bt}\cos(mx) \quad ; \quad y = 0 \qquad (9)$$

$$2\mu_1 u_y = \frac{\partial\phi_1}{\partial y} = 0 \quad ; \quad y = 0 \qquad (10)$$

Similarly, for the half-plane y<0, we can use the potential

$$\phi_2 = \frac{2\mu_2\alpha_2(1+\nu_2)k_2A_2}{(1-\nu_2)b}\{e^{-a_2y} + \frac{a_2}{m}e^{my}\} \cdot e^{bt}\cos(mx) \qquad (11)$$

which also gives $\sigma_{xy} = u_y = 0$ on y = 0 and

$$\sigma_{yy} = -\frac{2\mu_2\alpha_2(1+\nu_2)A_2 m}{(1-\nu_2)(m-a_2)} e^{bt}\cos(mx) \; ; \quad y = 0 \qquad (12)$$

5 THE CONTACT PROBLEM

We require that the half-planes make frictionless contact at the interface y = 0 and hence that

$$u_{y1} = u_{y2} \qquad y = 0 \qquad (13)$$

$$\sigma_{xy1} = \sigma_{xy2} = 0 \qquad y = 0 \qquad (14)$$

$$\sigma_{yy1} = \sigma_{yy2} \qquad y = 0 \qquad (15)$$

We also require that the heat flux be continuous at the interface and hence

$$q_{y1} = q_{y2} \qquad y = 0 \qquad (16)$$

The particular solution of the previous section already satisfies conditions (13,14) and it can be made to satisfy (16) by defining the constants A_i in terms of a new constant, A, such that

$$A_1K_1a_1 = A_2K_2a_2 = A \qquad (17)$$

where K_i are the thermal conductivities of the materials.

However, the solution does not satisfy (15). We must therefore superpose an isothermal solution corresponding to each half-plane being loaded by a sinusoidal normal traction.

It is readily verified (for example by using a potential of the form $C\exp(-my)\cos(mx)$ in the solution of Green and Zerna [12], section 5.7), that a normal traction

$$\sigma_{yy} = B_1\cos(mx) \tag{18}$$

on the surface $y = 0$ of the half-plane $y>0$ produces a normal surface displacement

$$u_y = -\frac{B_1(1-\nu_1)}{m\mu_1}\cos(mx) \tag{19}$$

We superpose this solution on the thermoelastic field of equations (7-10) and a similar solution, with B_2 replacing B_1, for the half-plane $y<0$. To retain continuity of normal displacements at the interface (condition (13)), we require

$$\frac{B_1(1-\nu_1)}{\mu_1} + \frac{B_2(1-\nu_2)}{\mu_2} = 0 \tag{20}$$

and imposing condition (15) for the complete solution, we obtain

$$-\frac{2\mu_1\delta_1 Ame^{bt}}{(1-\nu_1)a_1(a_1+m)} + B_1 = -\frac{2\mu_2\delta_2 Ame^{bt}}{(1-\nu_2)a_2(m-a_2)} + B_2 \tag{21}$$

where the distortivity, $\delta = \alpha(1+\nu)/K$.

Equations (20,21) can be solved for B_i in terms of A and the results used to express the perturbations in the temperature difference, heat flux and contact pressure at the interface $y = 0$, in terms of the single constant, A. The expressions obtained are

$$\Delta T = T_2 - T_1 = A\left(\frac{1}{a_2K_2} - \frac{1}{a_1K_1}\right)e^{bt}\cos(mx) \tag{22}$$

$$\Delta q = q_y = Ae^{bt}\cos(mx) \tag{23}$$

$$\Delta p = -\sigma_{yy} = 4M\left(\frac{\delta_1}{a_1(a_1+m)} + \frac{\delta_2}{a_2(m-a_2)}\right) \cdot Ame^{bt}\cos(mx) \tag{24}$$

where

$$\frac{1}{2M} = \frac{(1-\nu_1)}{\mu_1} + \frac{(1-\nu_2)}{\mu_2} \tag{25}$$

6 PERTURBATION OF THE THERMAL RESISTANCE RELATION

To complete the solution, we linearize the equation defining heat conduction across the thermal contact resistance, R, for small perturbations about the steady-state.

The definition of the pressure dependent contact resistance, R, implies that

$$q_y = T^*/R(p) \tag{26}$$

where T^* is the temperature drop across the interface. Hence for small perturbations about the steady state, we have

$$R_0\Delta q + q_0\Delta R = \Delta T \tag{27}$$

Finally, noting that

$$R = R'\Delta p \tag{28}$$

we substitute for ΔT, Δq, Δp from equations (22-24) to obtain the characteristic equation

$$4MR'q_0m\left(\frac{\delta_1}{a_1(a_1+m)} + \frac{\delta_2}{a_2(m-a_2)}\right) + R_0 + \left(\frac{1}{K_1a_1} - \frac{1}{K_2a_2}\right) = 0 \tag{29}$$

Remembering that a_i is defined in terms of b through equation (4), this can be treated as an equation for the exponential growth rate, b.

7 IMPLICATIONS FOR STABILITY

The parameter, m, defines the spatial frequency of the sinusoidal perturbation in the horizontal direction. A random initial perturbation may be conceived as decomposed into a spectrum of such frequencies and if the exponential growth rate associated with any one of them has a positive real part, the steady-state solution with uniform pressure will be unstable.

The system shown in Figure 2 is periodic with wavelength $2\pi r$ and hence only admits the frequencies

$$m = n/r \tag{30}$$

where n is an integer. In particular, there is a minimum value, $m_1 = 1/r$.

In contrast, all positive values of m are admissible for the contact of two half-planes (Figure 1).

8 ONE BODY RIGID

We consider first the case where body 2 is a rigid perfect conductor, in which case equation (29) reduces to

$$\frac{4MKR'q_0\delta}{(1+a/m)} + R_0Km(a/m) + 1 = 0 \tag{31}$$

where we have dropped the suffices on the material properties of body 1 and on a_1.

This equation can be solved for a/m to give

$$a/m = -\frac{1}{2}\left(1 + \frac{1}{R_0Km}\right) + \left[\frac{1}{4}\left(1 - \frac{1}{R_0Km}\right)^2 - \frac{4MR'\delta q_0}{R_0m}\right]^{1/2} \tag{32}$$

All the physical parameters and m must be positive, but the contact resistance, R, falls with increasing pressure, so that $R'<0$. Also, the parameter a describes the exponential decay of the perturbation with y and hence must have a positive real part. We therefore conclude that equation (32) has physically meaningful solutions if and only if

$$- \frac{16MR'\delta q_0}{R_0 m} > -(1 - \frac{1}{R_0 K m})^2 \qquad (33)$$

in which case, the solutions for a/m will be real. In all the following discussion, it should be noted that R'<0 and hence expressions containing $(-R'q_0)$ will be positive for $q_0>0$.

Furthermore, these solutions will only correspond to exponentially growing perturbations if a>m and hence

$$- \frac{2MR'\delta q_0}{R_0 m} > 1 + \frac{1}{R_0 K m} \qquad (34)$$

Thus, instability only occurs when the heat flow is directed into the deformable body ($q_0>0$) and exceeds a certain critical value. A similar conclusion was reached for the one-dimensional problem of an elastic rod making contact with a hot rigid wall [5].

The condition (34) also shows that the longer wavelength disturbances (smaller values of m) become unstable first. For the configuration of Figure 1, all wavelengths are admissible and hence instability occurs if

$$-2MR'K\delta q_0 > 1 \qquad (35)$$

For the two-cylinder configuration of figure 2 (here cylinder 2 could be replaced by a rigid plane), the lowest admissible value of m is 1/r and hence

$$-2MR'K\delta q_0 > 1 + \frac{R_0 K}{r} \qquad (36)$$

for instability.

If the heat flux, q_0 is regarded as given, condition (34) can be interpreted as defining a maximum value of m for instability, which is

$$m_{cr} = - \frac{2MR'\delta q_0}{R_0} - \frac{1}{R_0 K} \qquad (37)$$

All perturbations of longer wavelength will then have positive exponential growth rates, but the growth rate is a function of m and has a maximum at some value, m* where $0<m^*<m_{cr}$. Figure 4 shows the dependence of m* on the heat flux q_0. A very similar behaviour is observed in the sliding contact of elastic solids with frictional heat generation [13] where the critical wavelength is found to dominate the transient process. However, in the present system, the non-linearity of the contact resistance-pressure relation would be expected to modify the behaviour of the transient process, once the perturbation had grown to a significant magnitude.

9 TWO DEFORMABLE MATERIALS

In the preceding section, the only physically admissible solutions corresponded to real eigenvalues for the exponential growth rate, b. It follows that roots of equation (31) can only enter the unstable domain through the origin and hence that the stability boundary is defined by the condition, b = 0.

When both materials are deformable, the behaviour is more complex. If we examine first the condition, b = 0, we find that $a_1 = -a_2 = m$ and equation (29) reduces to

$$2MR'(\delta_1-\delta_2)q_0 + R_0 m + (\frac{1}{K_1} + \frac{1}{K_2}) = 0 \qquad (38)$$

which has solutions only if

$$q_0(\delta_1-\delta_2) > 0 \qquad (39)$$

i.e. if the heat flow is directed into the more distortive material. This conclusion accords with our knowledge of the steady-state behaviour of thermoelastic contact, where multiple solutions are generally observed for this direction of heat flow [5,6,8]. In particular, we note that Comninou and Dundurs' solution with periodic contact and separation zones for the problem of Figure 1, provides a steady-state solution towards which an unstable sinusoidal perturbation can grow.

However, a closer examination of equation (29) shows that (39) is not a necessary condition for real positive values of b to occur. Suppose we arbitrarily choose two materials which have equal distortivities, but different diffusivities, such that $k_1>k_2$. The first term in (38) is then zero for all values of q_0 and the other terms are always positive, so we conclude that there are no solutions of (29) with infinitessimally small real values of b. However, if b is not infinitessimal, the difference in diffusivities ensures that $a_1<-a_2$ (see equation (4)) and hence that the first term in equation (29) is negative for positive q_0. The remaining terms in (29) are positive and hence we conclude that there must be some value of q_0 which will satisfy the characteristic equation for any given non-zero value of b.

In mathematical terms, this means that, as q_0 increases, the complex zeros of equation (29) pass into the positive half-plane across the imaginary axis instead of through the origin, giving exponentially growing oscillatory solutions. At higher values of q_0, a zero is obtained at a finite point on the positive real axis, which then bifurcates, one branch approaching zero and the other infinity asymptotically.

Since we have taken the distortivities to be equal, the steady-state thermal distortion of the two bodies will be complementary - i.e. the local expansion of body 1 will exactly conform with the local contraction of body 2 [14] - and hence the heat flow will have no effect on the contact problem. It follows that the classical uniqueness theorem for isothermal elasticity applies and that the steady-state solution with uniform pressure and uniform heat flux across the interface is unique.

However, we have just demonstrated that, with a sufficiently high value of q_0, there is a perturbation with a real positive exponential growth rate. Thus, we reach the remarkable conclusion that there are conditions for which the system has a unique steady-state solution which is unstable.

We must presume that the growth of the perturbation will eventually be curtailed by the non-linearity of the contact resistance relation or by separation, but since there is no other steady-state solution, the system must settle into an oscillatory state. The possibility of oscillatory behaviour of thermoelastic contact has been discussed ever since difficulties with existence were first discovered with the classical boundary conditions [15]. Indeed, Clausing [16] reports slow periodic variations in experimental measurements of thermal contact resistance which may be attributable to this mechanism. A linear perturbation analysis as developed here cannot be used to draw conclusions about the long term behaviour of the system, but a numerical treatment of the corresponding transient contact problem is in progress and will be reported later.

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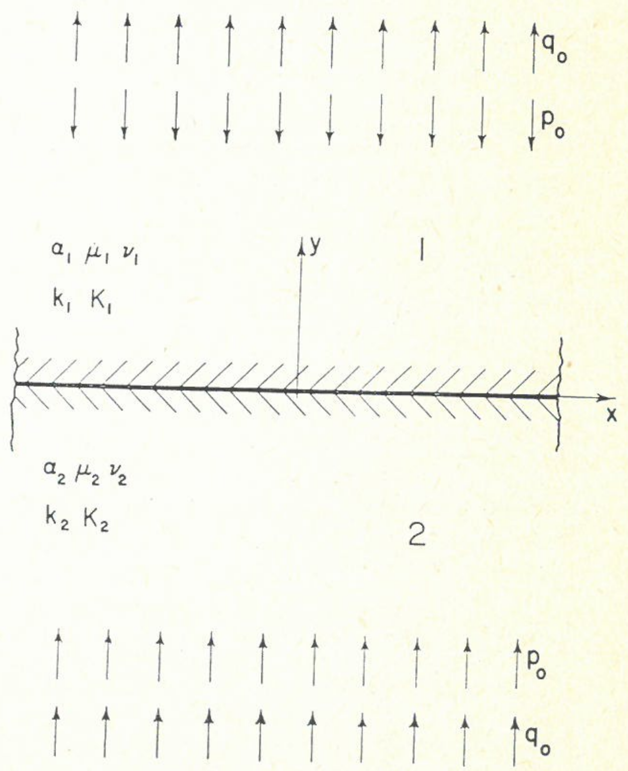

Fig 1 The two dissimilar half-planes, $y>0$, $y<0$, pressed together by a uniform pressure, p_o and transmitting a uniform heat flux, q_o

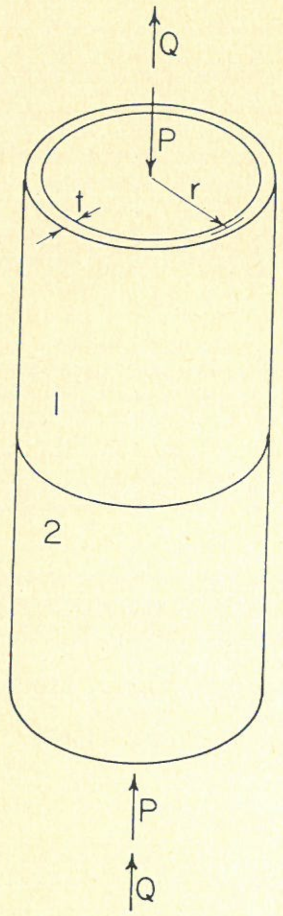

Fig 2 Contact of two thin-walled cylinders on an end face

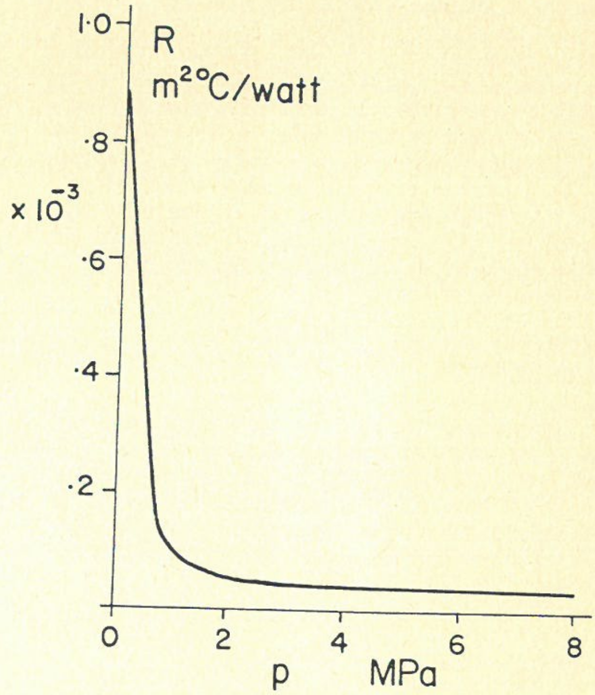

Fig 3 Typical experimental results for the variation of thermal contact resistance with pressure

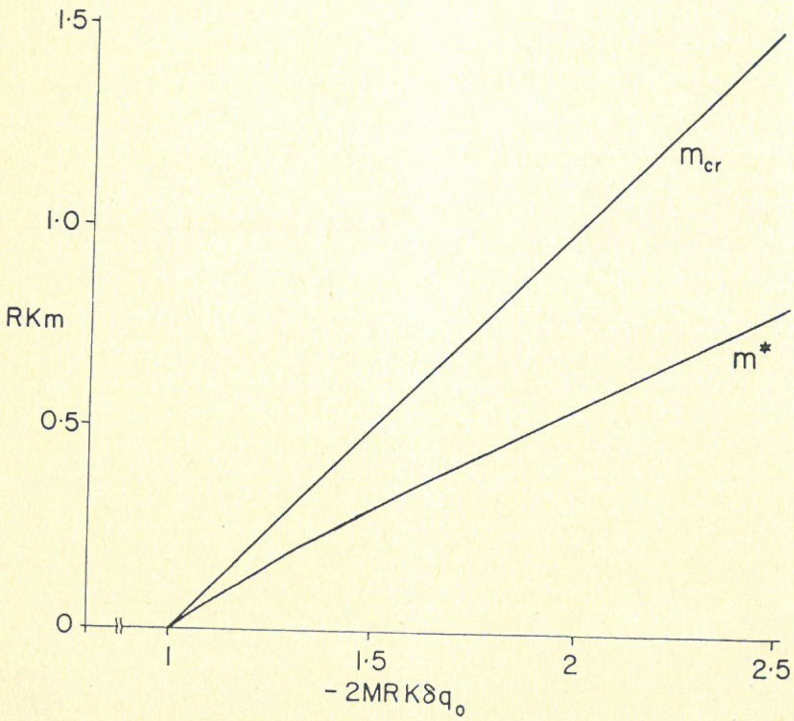

Fig 4 Effect of heat flux, q_0, on the frequency of the most rapidly growing perturbation, m^*, and the maximum frequency for instability, m_{cr}