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The application of asymptotic solutions to characterising the process zone in almost complete frictional contacts

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Abstract

The recently developed method of embedding the solution for a semi-infinite flat and rounded punch ('inner' asymptote) into the solution for a semi-infinite square-ended punch ('outer' asymptote) in order to solve a number of notionally complete contact problems, where in practice, a finite radius is present is extended to cases where friction arises. The first case considered is when the slip zone is large compared with the size of the process zone, and extends into the flat portion of the contact. The second problem is when the slip zone lies wholly within the curved portion of the contact. Here, the inner asymptotic solution is modified to allow for partial slip providing a *universal* solution for this class of problem, and a comparison with the outer solution is made.

These techniques show that the conditions prevalent in the process zone of any notionally complete frictional contact can be accurately represented by recourse to the inner and outer asymptotes, thereby eliminating the need to solve the *full* finite contact problem explicitly; only the corresponding complete, adhered problem must be solved numerically.

1. Introduction

Fretting damage is often found adjacent to the edges of notionally 'complete' contacts. For example, if a flat-ended indenter is pressed against a block of similar material and an oscillatory shearing force applied, a narrow ring of 'cocoa' will subsequently be founded around the periphery. A simple idealised analysis of the problem assuming a rigid circular punch pressed onto an incompressible half-space suggests a different response: the contact pressure, $p(r)$, is of the form

$$\frac{a^2 p(r)}{P} = \frac{1}{\pi\sqrt{1-(r/a)^2}}, \tag{1}$$

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where P is the applied load and a is the radius of the indenter. The shearing traction distribution corresponding to a rigid-body tangential displacement of the surface is of the same form, and it follows that if the shearing force, Q, is less than fP, where f is the coefficient of friction, there will be adhesion everywhere. When the applied shearing force reaches the limiting value slip ensues everywhere, and there is, apparently, no stable partial slip regime. Of course, one idealisation which is being made is in the choice of elastic constants of the contacting bodies, which must be done in order to apply half-space theory rigorously, and to ensure that the direct and shearing components of traction are uncoupled. A numerical solution of the equivalent problem with elastically similar bodies will give rise to a very narrow band of slip, but this would be significant only very close to the sliding state. A second phenomenon of practical relevance is the inevitable presence of a slight radius around the periphery of the indenter, and it is the influence of this which we wish to explore in the present paper, using a development of recent asymptotic solutions for contact problems.

The use of asymptotic ideas to obtain efficient solutions for the stress state adjacent to the edge of complete or almost complete contacts has been developed in recent years, using, as a basis, the classical solutions for sliding wedges developed by Gdoutos and Theocaris (1975) and Comninou (1976). These solutions, which include the possibility of allowing for elastic mismatch, apply whenever the contact is truly complete, and have been applied to fretting problems successfully (Mugadu et al., 2002; Mugadu and Hills, 2002). One obvious question which arises when using this technique is how big a finite edge radius may be before the singular solution becomes invalid. This has been answered in the case of frictionless contacts by employing an asymptotic solution in the form of a semi-infinite flat and rounded punch (Sackfield et al., 2003). The paper should be consulted for the full details of the philosophy behind the method, but the essentials emerge from a consideration of the following four problems: (i) a notionally sharp square-ended finite rigid punch (in general of arbitrary planform). (ii) A punch of the same general planform but incorporating a very small edge radius, R. A solution may be found to problem (ii) using the solution to problem (i) *alone*, but employing the two following asymptotic solutions, of a completely general nature, and with exact closed forms, as a vehicle to move from one to the other.

(1) A semi-infinite square-ended rigid punch, and we shall denote this the 'outer' asymptotic solution. The contact pressure distribution for this configuration may be written in the form

$$p(x_1) = \frac{K^*}{\sqrt{x_1}}, \tag{2}$$

where x_1 is a coordinate measured from the contact edge, and K^* is a scaling multiplicative constant, analogous to the role played by the stress intensity factor in fracture mechanics. In the neighbourhood of the corner of the sharp punch (problem (i)) the contact pressure varies in this way, and hence the value of the generalised stress intensity factor, K^*, may be found by matching the two solutions in this region.

(2) The second asymptotic solution ('inner' solution) is that for a radiused semi-infinite flat and rounded punch, Fig. 1(a). The contact pressure distribution for this problem may be found from that for the *finite* flat-and-rounded punch Ciavarella et al. (1998). In the coordinate set shown the contact for a punch having a contact half-width b is given by

$$p(x_1) = -\frac{E^*}{2\pi R}\sqrt{x_1\left(1-\frac{x_1}{2b}\right)}\int_0^d \frac{(s-d)(s/b-1)\,\mathrm{d}s}{\sqrt{s(1-s/2b)}[x_1 - s + (s^2 - x_1^2)/2b]} \quad x_1 > 0, \tag{3}$$

where $E^* = E/(1-\nu^2)$, E being Young's modulus and ν Poisson's ratio. If, now, the limit $b \to \infty$ is taken and the scaling constant K^* introduced, defined by

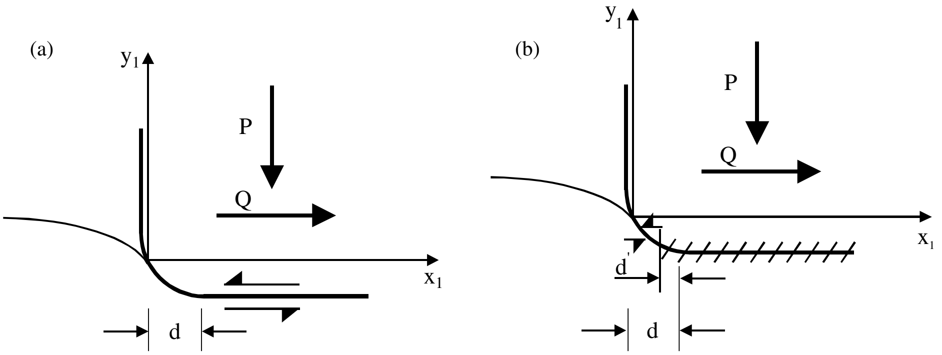

Fig. 1. Schematic of a semi-infinite flat and rounded punch in: (a) full sliding and (b) partial slip.

$$K^* = -\frac{2E^*\sqrt{d^3}}{3\pi R}, \tag{4}$$

we find that, after some algebra

$$p(x_1) = \frac{3K^*}{4\sqrt{d^3}}\left[2\sqrt{x_1 d} + (x_1 - d)\ln\left|\frac{\sqrt{d}-\sqrt{x_1}}{\sqrt{d}+\sqrt{x_1}}\right|\right]. \tag{5}$$

If $x_1/d \gg 1$, i.e. if the point under consideration is sufficiently remote from the contact edge, the pressure varies in the same form as that implied by Eq. (2), whilst, if $0 < x_1/d \ll 1$, i.e. the observation point is very close to the edge of the contact, the pressure varies in the form

$$p(x_1) = \frac{3K^*\sqrt{x_1}}{d}. \tag{6}$$

It may be seen that, if the 'inner asymptote' (the solution for the semi-infinite radiused punch) is scaled so that it matches the 'outer asymptote' (the solution for the semi-infinite square ended rigid punch) at large values of x_1, (and this is done simply by matching the values of K^*) the solution to the finite, almost complete problem (problem (ii)), may be found without further calculation. Implementation of this result requires the use of the contact law for the inner asymptote, which after some manipulation to interchange the dependent and independent variables is given by

$$d = \sqrt[3]{\frac{9\pi^2 R^2 K^{*2}}{4E^{*2}}}, \tag{7}$$

where R is the radius of the curved portion of the contact.

The intention in the present paper is to extend these results to the case when frictional effects are also present. First, modified forms of the asymptotic solution for the semi-infinite rounded punch will be found for both locally sliding and local partial slip conditions, and these will be employed to solve a range of frictional contact problems involving notionally complete contacts. In all cases it will be possible to determine the stress state adjacent to the contact edge, and therefore to establish the nature of the process zone where cracks initiate, and to do so without having to solve for the state of stress in the actual body; this will be provided by the asymptotic solution. Further, it will not be necessary to solve the nearly complete contact problem in order to determine the stick/slip distribution. This may be found from the corresponding finite contact problem which is truly complete in nature (when the slip zone encompasses at least part of the flat portion of the contact), whilst in other cases even this may not be needed: if the stick zone persists into the curved part of the contact face the extent of the slip region may be found from the asymptotic solution itself, and only the adhered form of the finite, complete contact problem will be needed.

The practical application of this solution is to the characterisation of the process (plastic) zone adjacent to the contact edge, where cracks nucleate. Its power is that, in cases of macroscopic partial slip, all local detail may be included trough the asymptote.

2. Asymptotic solution

The first stage in extending the technique is simply to extend the asymptotic solution itself. The problem to be solved is shown in Fig. 1(a). It shows the same semi-infinite flat and rounded punch subject to both normal and shearing loads, the ratio between the two being the coefficient of friction, f. Note that the sliding asymptotic solution generated in this way will turn out to have applications in both macroscopically sliding finite contact problems, and to those suffering partial slip, but where the slip region engulfs a large region of the interface and penetrates significantly into the flat region of the punch face. There is little which needs to be added to the solution presented in the Introduction for a punch giving rise to direct tractions alone, save that the shearing traction, $q(x_1)$, is everywhere equal to $\pm f p(x_1)$. Also, the Muskhelishvili potential for this problem is determined in the usual manner from the definition

$$\Phi(z_1) = \frac{1 - \mathrm{i}f}{2\pi\mathrm{i}} \lim_{b\to\infty} \int_0^b \frac{p(\xi)\,\mathrm{d}\xi}{\xi - z_1}, \tag{8}$$

where $z_1 = x_1 + \mathrm{i}y_1$. Substituting in from (5) we find

$$\Phi(z_1) = \frac{3K^*}{4\sqrt{d^3}} \frac{(1 - \mathrm{i}f)}{2} \left[2\sqrt{z_1 d} + (z_1 - d) \ln \left| \frac{\sqrt{z_1} - \sqrt{d}}{\sqrt{z_1} + \sqrt{d}} \right| \right]. \tag{9}$$

We turn, now, to the asymptotic form when the stick region extends into the curved part of the contact, as shown in Fig. 1(b). For this case we may write down the resulting shear traction distribution using Ciavarella's theorem (Ciavarella, 1998a,b): we need to superimpose a second shear traction distribution (a perturbation, $q'(x_1)$), present over a reduced region, the new stick zone, on the sliding shearing traction distribution. The perturbation has the same form as the contact pressure distribution, but is due to a smaller applied load, P'

$$q'(x_1') = \frac{3K^{*\prime} f}{4\sqrt{d'^3}} \left[2\sqrt{x_1' d'} + (x_1' - d') \ln \left| \frac{\sqrt{d'} - \sqrt{x_1'}}{\sqrt{d'} + \sqrt{x_1'}} \right| \right], \tag{10}$$

where x_1' and d' are here measured from the slip–stick transition point, and $K^{*\prime}$ is due to the adjusted 'corrective' load P'. Eq. (10) is scaled appropriately in magnitude by using the contact law (Eq. (7)), and from the application of tangential equilibrium, we may determine the size of the stick region, d', as

$$\frac{d'}{d} = \left(1 - \frac{Q}{fP}\right)^{2/3}. \tag{11}$$

This then allows us to write Eq. (10) in terms of the global parameters. Fig. 2(a) shows a plot of the resultant shear traction distribution as a function of Q/fP, and Fig. 2(b) shows the size of the stick zone, also as a function of Q/fP. Included in the latter plot, for comparison purposes, is the corresponding stick zone size for a *finite* flat-and-rounded punch Ciavarella et al. (1998). It may be seen that the stick zone size in the finite contact is only very slightly different from the semi-infinite contact if $a/b > 0.9$, where a is the flat portion half-width and b is the contact half-width.

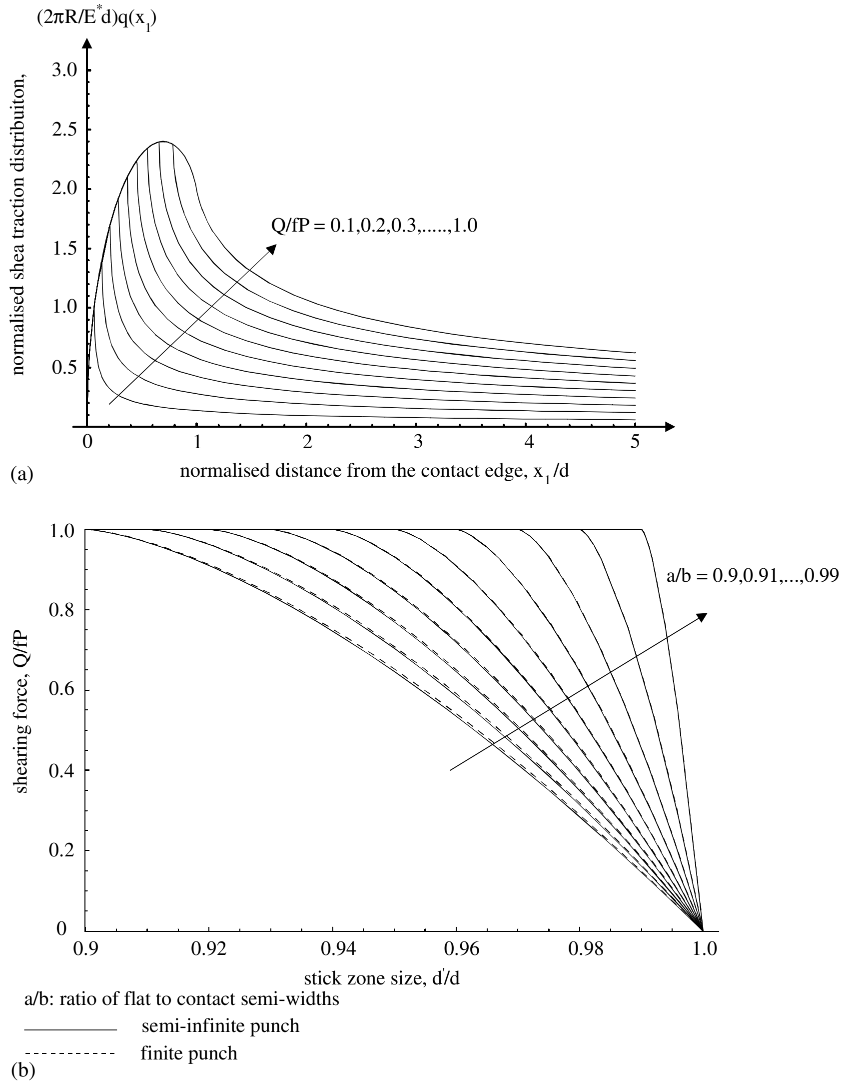

Fig. 2. (a) Plot showing the shear traction distribution as a function of Q/fp for a semi-infinite flat and rounded punch and (b) plot showing the stick zone size as a function of Q/fp for the semi-infinite and finite flat and rounded configurations.

3. Applications

The solutions developed in the previous section will be applied to three classes of contact, and these are shown schematically in Fig. 3. It is emphasised again that the techniques described may be applied to *any* contact which is notionally *complete*, locally two-dimensional in nature, and exhibits a square-root singularity. The classical square-ended finite contact is used purely as a vehicle to display the results, and similar procedures may be applied to punches with internal voids, or which are multiply-connected, or having a combination of abrupt and smooth edges.

The three classes of responses which we identify are: (a) when the macroscopic punch is in sliding, and the sliding asymptotic solutions may be applied directly, (b) when the contact is in partial slip but the slip

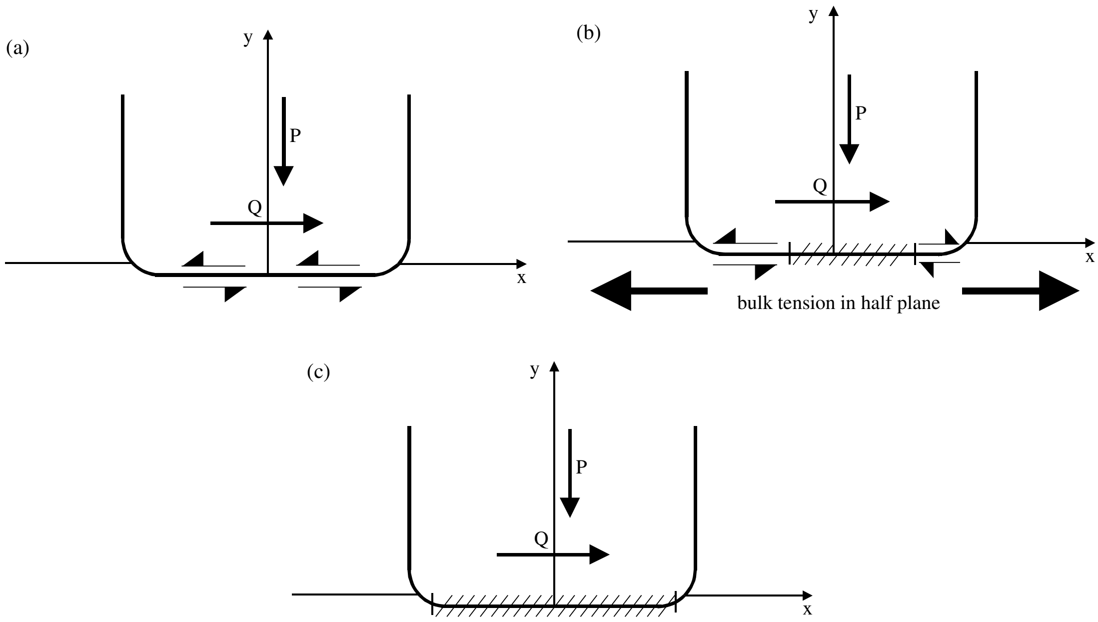

Fig. 3. Schematic of a flat punch with rounded edges under: (a) full sliding; (b) partial slip with the slip zone extending into the flat portion of the contact, and (c) partial slip with the slip zone in the curved region of the contact.

zone extends well into the flat portion of the punch face (here too the sliding asymptotic solutions may be applied), and (c) when the contact is in partial slip but the slip zone is contained wholly within the radiused portion of the contact profile, and here we compare the partial slip 'inner' asymptotic solution with the 'adhered' outer asymptotic solution. The solutions to these three classes of contact will now be presented in detail.

3.1. Sliding contact

The first problem we wish to consider is that of the sliding square-ended punch. It is assumed that the punch is rigid and the contacting half-plane is incompressible so that the direct and shear tractions are uncoupled, and each behaves in a square-root singular manner. In this, and subsequent problems, the first part of the procedure is to determine the macroscopic *sharp* contact pressure distribution, and then to collocate the semi-infinite square ended punch solution with the local contact edge pressure field, thereby giving the value of the generalised stress intensity factor, K^*. This part of the solution follows exactly the same procedure as that for the frictionless punch, described in Sackfield et al. (2003).

The next step varies from problem to problem, but here we simply record the Muskhelishvili potential for the asymptote, and hence deduce the stress field adjacent to the contact corner from which the local plastic or process zone may be found. The validity of the plastic or process zone as calculated from the singular field has two limits; an upper limit which corresponds to small-scale yielding in fracture mechanics, and where higher order terms associated with the presence of remote boundaries begin to have a significant effect on the shape and size of the process zone, and a lower limit, where the effects of the radius present at the contact corner start to be felt.

It will be recognised that the lower limit is determined by comparing the two asymptotic fields alone (the 'inner' and 'outer' solutions), and is therefore universal in nature, whereas the small scale yielding limit is geometry dependent, and therefore no general conclusions can be drawn. In the case of a *frictionless* contact, it proved impossible to determine explicitly conditions under which the elastic hinterland of the perfectly sharp and radiused semi-infinite solutions matched to within a given tolerance Sackfield et al. (2003). This was because, in the absence of friction, and given that the half-plane is assumed incompressible, the surface does not yield, so that all the plastic yield fronts converge at the contact edge. In light of this, it was not possible to determine a plastic zone contour which *completely* encompassed any given error contour. Therefore, we had to be content with displaying the implied shape and size of the process zones for the two problems, to highlight their similarity.

Here, however, it will transpire that a much stronger statement of the closeness of the two solutions is feasible, because it is possible to provide a very close comparison of the two semi-infinite fields rigorously. Fig. 4(a) displays this comparison for a sample coefficient of friction, viz. $f = 0.6$. The plot includes two sets of contours. One is the size and shape of the process zone for a given applied load, as implied by the inner solution, and we define a dimensionless load parameter, $\Lambda = K^*/k\sqrt{d}$, where k is the yield stress in pure shear. The other set of contours gives the explicit fractional difference (as a percentage) between the inner and outer asymptotic solutions using the von Mises' yield parameter, $(\sqrt{J_2}^{\text{outer}} - \sqrt{J_2}^{\text{inner}})/\sqrt{J_2}^{\text{outer}}$, as a measure of the stress state. Of course the discrepancy between the two solutions increases as the contact edge is approached, and the two solutions converge as the observation point becomes more remote. Thus, for example, to ensure that the elastic hinterland matches everywhere to within 5%, and with a coefficient of friction of $f = 0.6$, the minimum load which may be sustained is

$$\Lambda_{\min} = \left(\frac{2E^*K^{*2}}{3\pi Rk^3}\right)^{1/3} = 13.5, \tag{12}$$

where d has been replaced using Eq. (7) above. It is emphasised that this comparison is universal, in the sense that the outer asymptote may now be installed in *any* appropriate *finite* complete sliding punch edge, through appropriate choice of K^*.

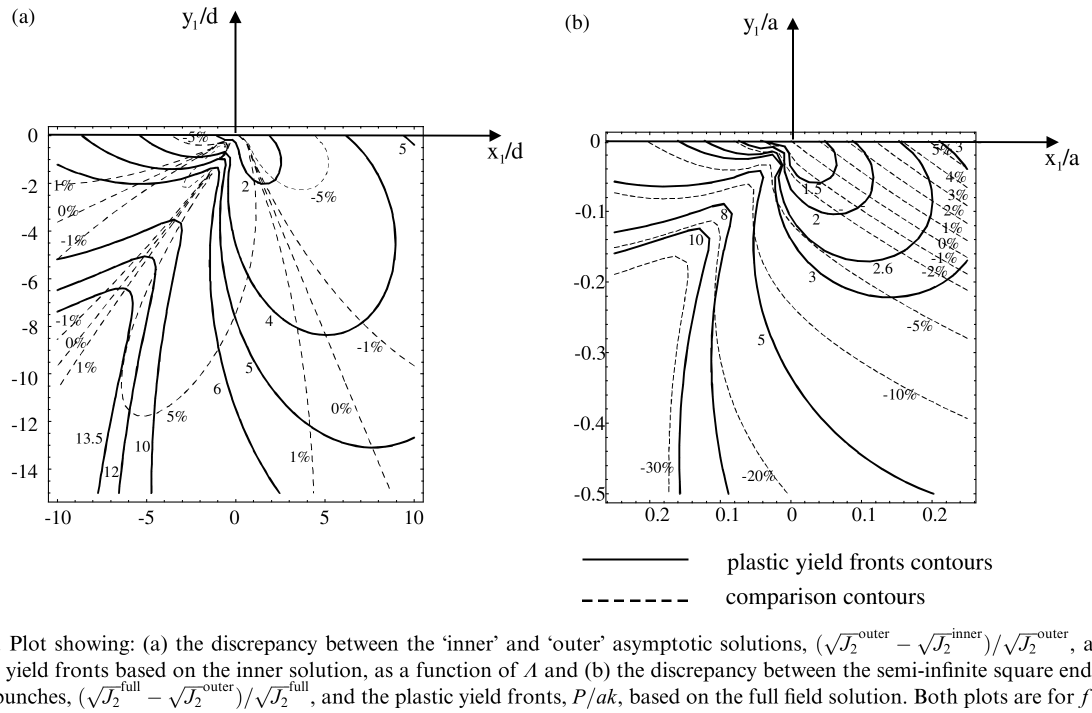

Fig. 4. Plot showing: (a) the discrepancy between the 'inner' and 'outer' asymptotic solutions, $(\sqrt{J_2}^{\text{outer}} - \sqrt{J_2}^{\text{inner}})/\sqrt{J_2}^{\text{outer}}$, and the plastic yield fronts based on the inner solution, as a function of Λ and (b) the discrepancy between the semi-infinite square ended and finite punches, $(\sqrt{J_2}^{\text{full}} - \sqrt{J_2}^{\text{outer}})/\sqrt{J_2}^{\text{full}}$, and the plastic yield fronts, P/ak, based on the full field solution. Both plots are for $f = 0.6$.

We turn, now, to a consideration of the upper load bound, or 'small scale yielding' limit. In order to do this we compare the solutions for the semi-infinite square ended punch and the finite square ended punch. As stated, this is geometry specific, and so here the measure of the load is the dimensionless ratio P/ak, where a is the punch half-width. Two families of contours are displayed in Fig. 4(b), and these are analogous to those used in Fig. 4(a); one family shows the location of the plastic yield fronts as implied by the elastic finite punch solution (full field solution), whilst the other shows the fractional discrepancy between the semi-infinite and finite punch solutions using the von Mises' parameter, $(\sqrt{J_2}^{\text{full}} - \sqrt{J_2}^{\text{outer}})/\sqrt{J_2}^{\text{full}}$, for a sample coefficient of friction, viz. $f = 0.6$. As an illustration in the use of this figure, if $f = 0.6$ then the upper load bound such that conditions of small scale yielding are satisfied to with a 5% tolerance is $P/ak = 2.6$. The two bounds may conveniently be displayed on the same figure. First, the lower bound solution is made specific to the square-ended punch problem by particularizing the stress intensity factor solution, i.e. here putting $K^* = P/\pi\sqrt{2a}$, after which the dimensionless quantity, Rk/E^*a, is defined with the aid of Eq. (7). The two bounding values on the load may then be compared directly, and these are shown in Fig. 5(a) for $f = 0.3$, and Fig. 5(b) for $f = 0.6$ and 0.9.

As an illustration in the practical use of the above results in predicting the upper and lower load bounds, consider the case of a nominally flat pad with $a = 5$ mm. Assume that the material has an elastic constant E^* of 112 kN/mm^2, a yield strength in shear, k, of 475 N/mm^2, and that the coefficient of friction is 0.6. If the pad radius were 10 μm, and the maximum tolerable discrepancy is 5% then $\Lambda_{\min} = 13.5$. The lower load bound is then found from the following equation:

$$P \geqslant \sqrt{\left(\frac{3Ra(\pi k \Lambda_{\min})^3}{E^*}\right)}. \tag{13}$$

With the quantities used for this example, we obtain $P_{\min} = 3309$ (N/mm). The upper load bound is found by noting from above that $P/ak = 2.6$ (for $f = 0.6$) so that $P_{\max} = 6175$ (N/mm). If, on the other hand, $f = 0.9$ then we have $\Lambda_{\min} = 7.0$ for a tolerable discrepancy of 5%, and this translates into a lower load bound of $P_{\min} = 1236$ (N/mm). For this coefficient of friction, $P/ak = 2.0$, and leads to $P_{\max} = 4750$ (N/mm).

These results for the lower load bound should be compared with those obtained in (Sackfield et al., 2002) where the lower load bound was found by comparing the *finite* flat-and-rounded punch and the 'outer' asymptote to give $P_{\min} = 3919$ and 1259 (N/mm) for $f = 0.6$ and 0.9, respectively. Clearly, the two approaches to predicting the lower bound load produce different results. This is to be expected since in Sackfield et al. (2002), the effect of higher order terms on the square-root singular stress field is to alleviate the stress state. It follows that comparing the two asymptotic solutions eliminates the influence of higher order terms remote from the contact edge. This becomes obvious when one notes that the region over which matching to a specified discrepancy is achieved is closer to the contact edge for larger values of f, i.e. smaller values of $\Lambda_{\min}$, and hence the influence of higher order terms becomes negligible.

Lastly, the current approach provides a lower bound to predicting the minimum allowable load. This clearly introduces a modest uncertainty in the solution, in the sense that an upper bound to the minimum load would be preferable. However, in practice, in the problems treated so far the discrepancy is small in comparison with the tolerance set for the asymptotic solution itself.

3.2. Partial slip: extensive slip region

The next problem to be considered is when the contact is suffering partial slip, and the slip region extends well into the flat part of the punch face. In practice, this situation is most often brought about when a tension is applied in the half-plane synchronously with the shearing force, and under conditions of pro-

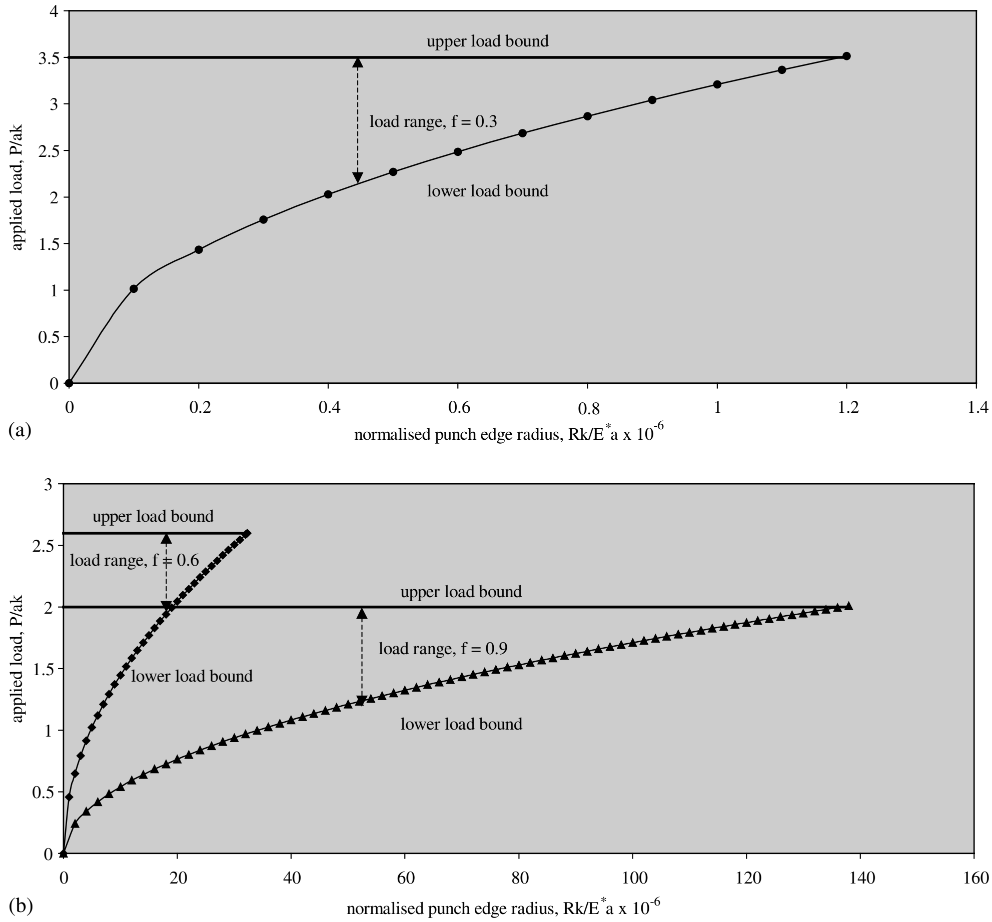

Fig. 5. (a) Plot showing the applied load, P/ak, against the punch edge radius, $Rk/E^{*}a$, when $f = 0.3$ and the tolerable discrepancy is 5% and (b) plot showing the applied load, P/ak, against the punch edge radius, $Rk/E^{*}a$, for a tolerable discrepancy of 5% when $f = 0.6$ and 0.9.

portional loading. This problem has been considered in some detail Navarro et al. (2003), but here, we shall simply reproduce the relevant results.

Note that the extent of the slip zone is determined by the ratio $\sigma_{o}a/Q$, where σ_{o} is the bulk tension, and that, although the macroscopic contact is in partial slip the region attached to the punch corner is in *sliding*, so that the appropriate asymptotic solution here is again one of sliding, as presented in the previous section. It follows that, ceteris paribus, the state of stress adjacent to the contact edge is independent of the magnitude of the shearing force, provided that the region over which the overall (finite) contact is suffering slip is much greater than the size of the domain over which the asymptotic solution is to be applied.

This raises the question of the quantities determining the bounds of the validity of the slipping asymptotic solution. Suppose that we use the sliding asymptotic solution to characterise a process zone which is ideally due to the sharp-edged solution. It follows that the lower bound for the acceptable load for this to be so is again determined by the comparison of the 'inner' and 'outer' asymptotic solutions, *exactly* as in the previous section, and no further qualification is needed. However, the upper bound to the load is now more restricted: in the sliding problem the feature which would cause a significant deviation from the semi-infinite solution ('outer' solution) is the presence of the far boundary of the punch. Here, it is the stick–slip boundary which is not properly represented in the sliding asymptote ('outer' solution). It follows that the 'small scale yielding' limit may now be found by comparing the sliding square-ended punch asymptotic stress state with the *finite* square-ended punch in partial slip.

Another feature of this solution which merits comment is the influence of the bulk tension on the stress state adjacent to the punch corner. There are parallels with the effect of the 'T-stress' in fracture mechanics, but in the presence of a finite radius at the contact edge, we would expect the effect to be more severe. We note that under these conditions the characteristic stress state becomes bounded as the punch corner is approached, and we therefore anticipate a greater influence on the range of loads which can be tolerated. The equivalent problem in fracture mechanics would be that of a rounded-end crack. If we were characterising the process zone simply by the singular square-ended solution, we might expect the bulk stress to produce only a moderate, short-range influence. However, at any finite distance from the singular point, the influence of the bulk tension would become significant. Additionally, an indication of the influence of the bulk tension can be found by comparing the direct stress parallel to the free surface of the half-plane in the absence of bulk tension with the magnitude of the bulk tension. Fig. 6 shows the comparison between the 'outer' asymptote and the *finite* square-ended punch in partial slip under the following conditions; $\sigma_o a/Q = 1$ so that the stick zone lies in the region $-0.273 \leqslant x \leqslant 1$, i.e. it is attached to the right hand edge, and with a coefficient of friction of $f = 0.6$. We note that the upper bound on the applied load such that the conditions of small scale yielding hold is given by $P/ak = 0.24$, where the tolerable discrepancy is set to 5%. This is significantly lower than the upper bound without bulk tension, viz. $P/ak = 2.6$ (Fig. 4(b)), confirming that in this case, the influence of the bulk tension is significant.

Lastly in this section we turn to the question of the effect of the edge radius on the location of the stick–slip interface. The thesis we defend is that the partial slip problem is very adequately represented by the square-ended punch problem, and that it is not necessary to determine the effect of the edge radius on the location of the stick–slip boundary. In general this phenomenon would be quite difficult to gauge, but the partial slip solution to the *finite* flat-and-rounded punch is available in this instance Ciavarella et al. (1998).

3.3. Partial slip: small zone of slip

In the absence of bulk tension, the application of a shearing force, Q, to the square-ended finite punch always leads to apparent adhesion everywhere, if $Q < fP$. On the other hand, the presence of a small radius at the edge of the contact will always lead to a very small region of slip, confined to a strip within the radiused portion of the contact. This may be analysed using the solution for the adhered outer asymptote together with the partial slip asymptote without recourse to further calculation. As before, K^* is found by collocating the 'outer' asymptote with the pressure distribution due the *finite* adhered square-ended punch. When once this is done the complete local stress state may be found simply from the asymptotic solution itself. As mentioned above, the upper bound to the domain of validity of this solution is again determined by geometrical considerations, as the asymptote correctly incorporates the presence of the entire stick region.

The question which arises is the magnitude of the process zone which may be appropriately represented by the 'outer' asymptotic solution to within a given tolerable discrepancy. As in the case of the frictionless punch, Sackfield et al. (2003), it is not possible to define a yield front based on the 'inner' solution that *completely* encompasses any given tolerable discrepancy. Therefore, we shall be content to simply display the yield fronts due to both asymptotic solutions so as to show their similarities. Before proceeding, it is noteworthy that there is some ambiguity in the relative lateral position of the two sets of contours, since we might (for example) identify the corner of the sharp punch with the end of the flat section of the rounded punch, or with some other point in the curved portion of the contact. Assuming the material to be homogeneous, failure may be influenced by the size and shape of the failure zone, but not by its lateral

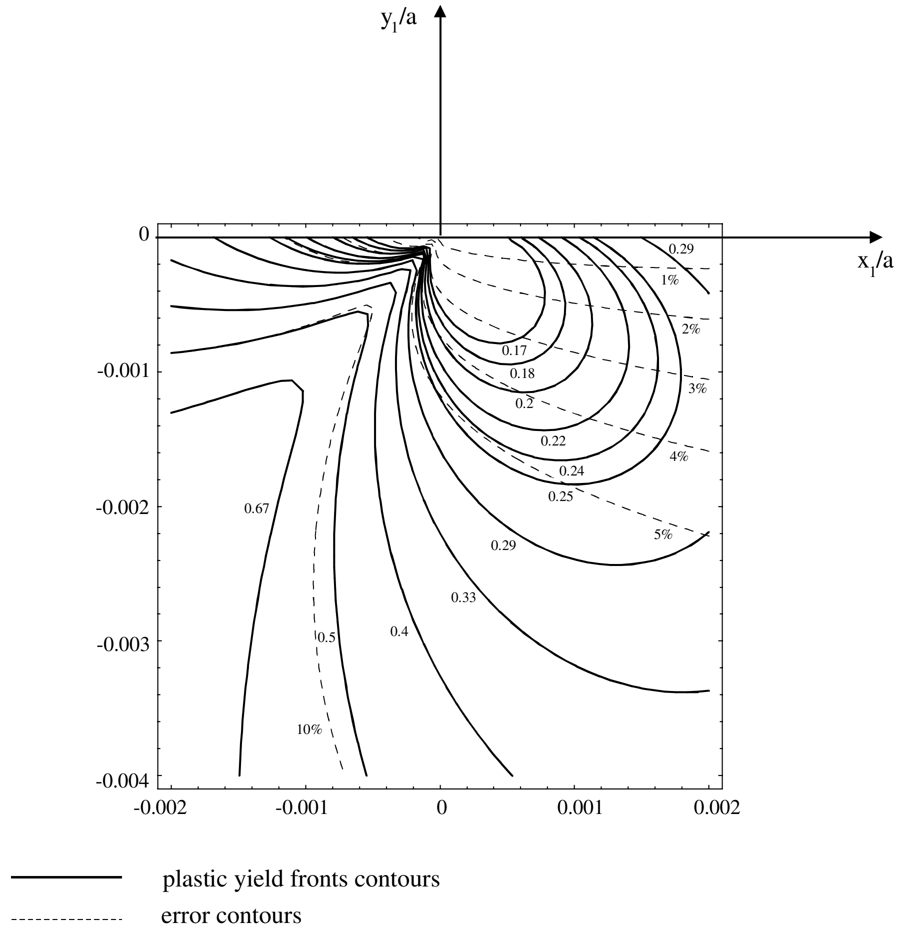

Fig. 6. Plot showing: (a) contours of the comparison between the finite square-ended punch with bulk tension present in the half-plane and the semi-infinite square-ended asymptotic solutions, $(\sqrt{J_2}^{\text{bulk}} - \sqrt{J_2}^{\text{asymp}})/\sqrt{J_2}^{\text{bulk}}$, when $\sigma_o a/Q = 1$, and (b) the plastic yield fronts based on the full field solution as P/ak. Both sets of contours correspond to $f = 0.6$.

position. We have therefore plotted Fig. 7 so as to achieve the best fit at large values of the loading parameter. These figures show plots of the yield fronts as defined by the two asymptotic solutions (the dashed contours correspond to the 'outer' solution) for a sample coefficient of friction, $f = 0.6$. Fig. 7(a) corresponds to the case when $Q/fP = 0$ so that no shearing tractions arise. This figure is similar to that shown in Fig. 5 in Sackfield et al. (2003), and we note that both asymptotic solutions are practically identical for

$$\left(\frac{2E^*K^{*2}}{3\pi Rk^3}\right)^{1/3} > 3.3. \tag{14}$$

In Fig. 7(b) we have $Q/fP = 0.05$, and we note that in this case the two solutions are practically identical for

$$\left(\frac{2E^*K^{*2}}{3\pi Rk^3}\right)^{1/3} > 6.7. \tag{15}$$

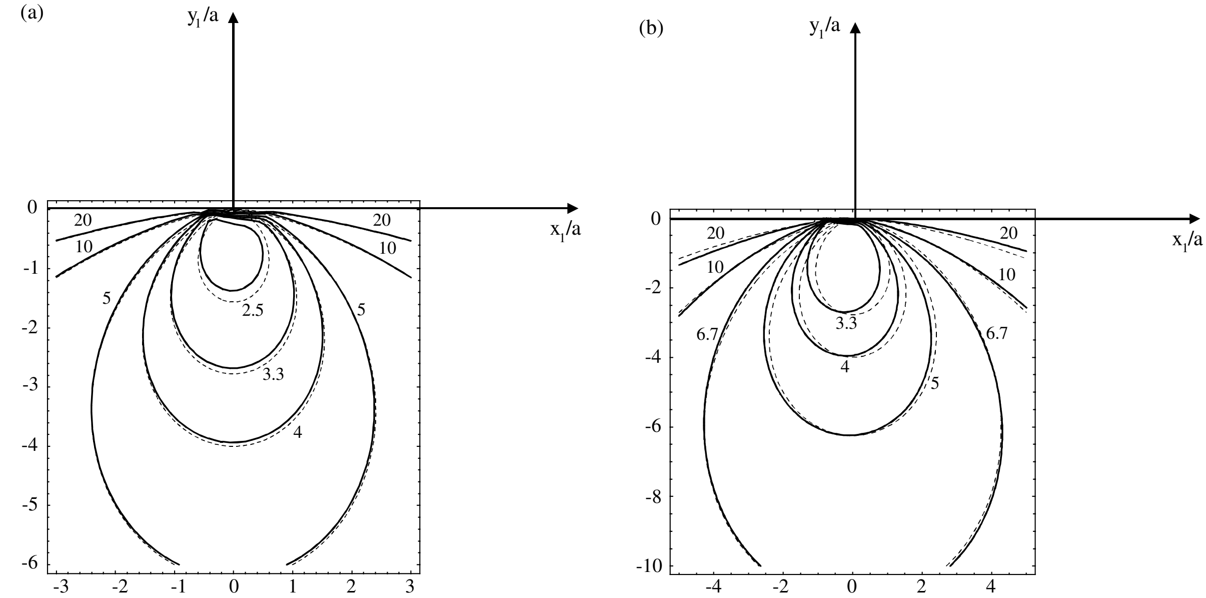

Fig. 7. Plots showing contours of the plastic yield fronts for the semi-infinite flat and rounded asymptote in partial slip with $f = 0.6$ (solid contours) and the 'adhered' semi-infinite square-ended asymptote (dashed contours) when: (a) $Q/fp = 0$ and (b) $Q/fp = 0.05$.

Note that in plotting Fig. 7, a greater lateral displacement (within the curved portion of the contact) was required for Fig. 7(b) than for Fig. 7(a). Therefore, as Q/fP increases, a larger load is required so that the two asymptotic solutions match to within a given tolerance.

4. Conclusion

The method of embedding the solution of a semi-infinite flat and rounded punch ('inner' asymptote) into the solution for a semi-infinite square ended punch ('outer' asymptote) has been extended to the case of nominally complete frictional contacts. It is emphasised that the comparison between these two solutions is universal, and holds for any nominally complete frictional contact exhibiting a square-root singularity. It was noted that the finite square-ended punch was used purely as a means of demonstrating the applicability of this approach, but that the finite punch could have had any planform. The comparison between the two asymptotic solutions sets the lower bound to the acceptable load such that the process zone conditions are dictated by the 'outer' asymptotic solution. The validity of the outer asymptote is found by recourse to the assumptions of small scale yielding, and is geometry dependent. This sets the upper bound on the acceptable load.

This approach was used extended to two nominally complete frictional contact problems, without the need to solve them explicitly. The first problem considered was when the slip zone is large, and extends into the flat part of the contact. This necessitates the presence of a bulk tension in the half-plane, whose influence was shown to substantially alter the upper bound to the acceptable load, whilst the lower bound remained unaltered. The second problem was when the slip zones lies wholly within the curved portion of the contact. Here, the inner asymptotic solution was modified to allow for partial slip, and a comparison between this solution and the *adhered* outer asymptotic solution was made.

The results presented in this analysis form the basis of a nested asymptotic procedure for defining the load range where a strictly square-root singular local elastic stress field continues to apply, under frictional conditions. This therefore provides a means of defining the process zone in terms of a single parameter, the generalised stress intensity factor.

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