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INTERFACE CRACKS

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Abstract—Properties of elastic solutions to interface crack problems are discussed in both the open formulation, in which the crack faces are assumed to be traction-free and interpenetration is permitted, and the unilateral formulation, where a contact zone is generally established adjacent to the crack tips. It is shown that if the contact zone is sufficiently small compared with the other dimensions in the problem, the unilateral solution can be approximated by embedding a universal contact field—characterized by two linear scaling parameters—within the surrounding asymptotic field of the open solution. Results for a plane crack are compared with analytical solutions due to Dundurs and Gautesen and the method is also used to predict the extent of the contact region for a penny-shaped interface crack in combined shear and tension.

1. INTRODUCTION

The problem of the stress and displacement field due to the presence of a crack at the interface between two dissimilar elastic media has attracted interest in the theory of elasticity since the earliest solutions were obtained by Erdogan [1] and England [2] in 1965, but the subject has recently seen a resurgence of attention because of applications to the fracture of composite materials.

The earliest solutions of the problem—following the corresponding solutions for cracks in homogeneous materials—were based on the assumption that the crack faces were traction-free and exhibited the familiar oscillatory singularity at the crack tip, in which the magnitude of the stresses approach infinity with $r^{-1/2}$ as the distance r from the crack tip tends to zero, but also the ratio $\sigma_{r\theta}/\sigma_{\theta\theta}$ oscillates with increasing frequency as the tip is approached. This feature was, at an early stage, regarded with some suspicion, but, more seriously, it was remarked that the crack opening displacement $u_y^+ - u_y^-$ also oscillates as $r \to 0$, implying an infinite sequence of regions in which the crack opening displacement is negative, i.e. in which the crack faces interpenetrate each other.

This paradox was resolved in the context of the theory of elasticity by Comninou [3], who replaced the traction-free boundary condition at the crack faces by the boundary conditions for frictionless unilateral contact, i.e. at any given point, either the crack faces are traction-free, or the crack opening displacement is zero, the extent of the contact and separation domains being determined by inequalities stating that contact tractions can only be compressive and crack opening displacements can only be positive (no interpenetration). Surprisingly, although the original traction-free solutions exhibit an infinite sequence of progressively smaller interpenetration zones, Comninou's solution shows that there is a single contact zone at each crack tip.

Since Comninou's original paper was published, solutions have been produced for several other interface crack problems involving contact zones [4–6] and, perhaps more significantly for the arguments of the present paper, it has been demonstrated that the unilateral contact formulation has a unique solution [7]. Also, Gautesen and Dundurs [8, 9] have developed more powerful analytical techniques for solving the resulting integral equations.

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2. SMALL-SCALE CONTACT CONDITIONS

If a plane interface crack is loaded predominantly in tension—for example, if the ratio of the far-field tensile stress to shear stress is greater than unity—the resulting contact zones are many orders of magnitude smaller than the crack length and in most cases will be smaller than the "process zone" surrounding the crack tip, within which the elastic solution cannot be regarded as a realistic description of the stress state. Of course, the central thesis of Linear Elastic Fracture Mechanics (LEFM) is that whatever complicated processes occur within this zone are completely determined by conditions in the surrounding elastic zone and, hence, that the surrounding field can be used to characterize the conditions at failure. Thus, if the open solution and the contact solution differ significantly only within the process zone, the difference is of little practical importance and the two solutions might be regarded as interchangeable. Rice [10] describes this condition as "small-scale contact conditions", by analogy with "small-scale yielding conditions" in elastic–plastic fracture, where the plastic zone is small compared with the crack dimensions.

Of course, the loading will not always satisfy these conditions—for example, the far-field loading may consist predominantly of shear, or may even involve compressive rather than tensile normal stresses. Furthermore, there are many important applications in which the interface and the crack are not plane—for example, a crack at the interface between a cylindrical fibre and a surrounding matrix—in which it is not a trivial matter to determine *a priori* whether the loading conditions will lead to small-scale contact conditions at both crack tips.

For this reason, the unilateral contact formulation of interface crack problems is more satisfactory than the open formulation, since, within the context of linear elasticity, it generates the physically correct solution to the problem regardless of the geometry and the loading conditions. When the contact solution and the open solution are, to all practical purposes, identical, either may be used whereas, when they differ significantly, only the contact solution can be regarded as correct.

However, we pay a price in adopting the contact formulation, since the governing inequalities make the problem non-linear and generally necessitate an iterative solution. Sometimes iteration can be avoided by regarding the extent of the contact zone as given and treating a ratio between loading parameters or a dimensionless material property as a dependent variable but, even then, the resulting boundary value problem is likely to be substantially more complicated than that for the open formulation.

There is, therefore, a strong motivation for using the open formulation whenever it is applicable, which, in turn, requires that we identify characteristics of the open solution which would indicate whether the restrictions it implies are satisfied. An informal way to do this is to determine the location of the furthest point from either crack tip at which interpenetration is predicted. Intuitively, if this is not small compared with the crack dimensions, we should expect significant differences between the open and contact solutions and we shall show in Section 5, below, that if this interpenetration length *is* small relative to the crack length, it bears a unique relationship to the extent of the contact zone in the contact solution.

3. PROPERTIES OF THE OPEN SOLUTION

The dominant singular term in the asymptotic field at an open crack tip on a material interface can be identified by expanding the length scale indefinitely in the vicinity of the tip, which is equivalent to determining the most general non-trivial solution to the problem of the semi-infinite interface crack between two dissimilar half-planes, with traction-free crack surfaces. This asymptotic technique was introduced by Williams [11] and has since been used extensively in the investigation of local fields at discontinuities in elasticity.

The interface can be taken to occupy the line $y = 0$, with the crack tip at the origin, the left half-line $x < 0$, $y = 0$ being bonded and the right half-line being cracked. Denoting the shear modulus and Kolosov's constant for the two half-planes by μ_i, κ_i respectively, where $i = 1$ refers to the upper half-plane, $y > 0$, and $i = 2$ to $y < 0$, we find [10] that the stresses on the bonded part of the interface can be written in the form

$$(\sigma_{yy} + i\sigma_{xy}) = Kr^{-1/2}\left(\frac{r}{a}\right)^{i\varepsilon} \tag{1}$$

where ε is a dimensionless bi-material parameter defined by

$$2\pi\varepsilon = \log\left(\frac{1+\beta}{1-\beta}\right) \tag{2}$$

and

$$\beta = \left[\frac{\kappa_1 - 1}{\mu_1} - \frac{\kappa_2 - 1}{\mu_2}\right] \Big/ \left[\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2}\right] \tag{3}$$

is Dundurs' parameter [12]. The corresponding expression for the discontinuity in displacement across the crack is

$$\Delta u_y + i\Delta u_x = (u_y + iu_x)_1 - (u_y + iu_x)_2 = \frac{\left[\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2}\right] K\left(\frac{r}{a}\right)^{i\varepsilon} r^{1/2}}{2(1 + 2i\varepsilon)\cosh(\pi\varepsilon)} \tag{4}$$

and

$$\gamma = \left(\frac{\frac{1}{\mu_1} + \frac{1}{\mu_2}}{\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2}}\right)$$

In these expressions, K is a complex number depending on the applied loading which plays a role analogous to that of the two stress-intensity factors, K_{I}, K_{II}, in the analysis of cracks in homogeneous materials. Notice also that we have introduced a length scale, a, because the imaginary part of the exponent on r appears in trigonometric functions when real and imaginary parts are separated, and these are only meaningful with dimensionless arguments. However, we emphasize that, notwithstanding the length scale, the asymptotic solution is self-similar.

The oscillatory nature of the asymptotic fields becomes apparent when we separate the real and imaginary parts of Eqns (1) and (4). In particular, we can define an angle θ_s, such that

$$\theta_s \equiv arg(\sigma_{yy} + i\sigma_{xy}) = arg(K) + \varepsilon\log\left(\frac{r}{a}\right) \tag{5}$$

from Eqn (1).

Clearly θ_s takes all possible values in the range $-\infty < \theta_s < \infty$ as r increases from zero to infinity. In fact, the complex function $\theta = arg(\sigma_{yy}(r) + i\sigma_{xy}(r))$ plots out as a logarithmic spiral centered on the origin which circles the origin an infinite number of times and cuts all straight radial lines from the origin at the same angle, *viz.* $\tan^{-1}(\varepsilon)$.

A curious feature of the asymptotic field is that all possible fields can be mapped into each other by an appropriate change in length scale and a linear multiplier. For example, we can always choose the length scale, a, so as to make K real and positive. To demonstrate this, we introduce a new length scale, a_1, in Eqn (5) obtaining

$$\theta_s = arg(K) + \varepsilon\log\left(\frac{r}{a_1}\right) + \varepsilon\log\left(\frac{a_1}{a}\right). \tag{6}$$

We can now choose to define a_1 such that

$$arg(K) + \varepsilon\log\left(\frac{a_1}{a}\right) = 0 \tag{7}$$

in which case Eqn (1) can be written

$$(\sigma_{yy} + i\sigma_{xy}) = K_1 r^{-1/2}\left(\frac{r}{a_1}\right)^{i\varepsilon} \tag{8}$$

where $K_1 = |K|$ and is a real positive constant.

Thus, no clear distinction can be made between shear and tensile fields, since what appears as a tensile field on a length scale a_1 will appear as a shear field on a scale of Ca_1 and indeed as a *compressive* field on a scale of $C^2 a_1$, where

$$C = \exp(-\pi/2\varepsilon). \tag{9}$$

Notice, also, that the asymptotic field can be linearly mapped onto itself under a scale length change in the ratio C^4. However, it must be immediately noted that the ratio C is always exceedingly small. The bimaterial constant ε cannot exceed 0.175 for material combinations with positive Poisson's ratios and for most practical combinations will be significantly lower. Thus, the maximum possible value of C is about 0.000126 and there will be many cases where a reduction of length scale of C or C^2 will carry us from the scale appropriate to the crack dimensions into a region contained within the process zone, where the elastic field is no longer assumed to be correct. In such cases, the asymptotic field can be identified as predominantly tensile or predominantly shear, etc. but no precise equivalent can be defined to the concept of mode-mixity, defined, for example, as the ratio K_I/K_{II} for homogeneous materials.

Rice [10] has suggested that the concept of mode-mixity could be reintroduced if a specific length scale, a, could be accepted as a convention. The complex constant K would then be uniquely defined and could be interpreted such that $K = K_I + iK_{II}$. We should note, incidentally, that as in the case of a crack in a homogeneous material, the energy release rate depends only on the magnitude of K and not on its argument, being given by

$$G = \left[\left(\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2}\right)\pi K\bar{K}\right] \Big/ (8\cosh^2(\pi\varepsilon)). \tag{10}$$

From an experimental fracture mechanics perspective, it would clearly be possible to conduct experiments under different far-field loading conditions (chosen so as to ensure small-scale contact conditions) and plot the results as a failure surface in K_I vs K_{II} space as so defined. In fact, the effect of changing the length scale on such a plot would simply be to rotate the failure surface about the origin through the (generally small) angle $\varepsilon\log(R)$, where R is the ratio between the two length scales. It is interesting to note, however, that an indication of a significant mode-mixity effect in such a plot could also be interpreted as a sensitivity to length scale in the failure mechanism. Indeed, it would also be possible to characterize the failure condition in terms of a linear multiplier such as $|K|$ and the length scale a_1 at which the field appears as purely tensile.

4. THE POINT OF FIRST INTERPENETRATION

Equation (4) can be used to determine the regions in which the open solution predicts interpenetration and this information is useful in estimating, often with considerable accuracy, the extent of the contact zone in the unilateral formulation. Defining a new angle θ_i by the relation

$$\theta_i = arg(\Delta u_y + i\Delta u_x) \tag{11}$$

we have

$$\theta_i = arg(K) + \varepsilon\log\left(\frac{r}{a}\right) - \arctan(2\varepsilon) \tag{12}$$

from Eqn (4).

Interpenetration zones will then be defined by the condition $\cos(\theta_i) < 0$, i.e.

$$(2n - \tfrac{3}{2})\pi < \theta_i < (2n - \tfrac{1}{2})\pi \tag{13}$$

where n is any integer. This, in turn, defines the regions $a_L < r < a_R$ where

$$a_L = a\exp\left[\{(2n - \tfrac{3}{2})\pi - arg(K) + \arctan(2\varepsilon)\}/\varepsilon\right]$$
$$a_R = a\exp\left[\{(2n - \tfrac{1}{2})\pi - arg(K) + \arctan(2\varepsilon)\}/\varepsilon\right] \tag{14}$$

on the crack plane.

Equation (14) defines an infinite sequence of interpenetration zones as n takes all integer values, positive and negative. Thus, there is no final (in the sense of remote from the crack tip) interpenetration region in the pure asymptotic field, corresponding to the semi-infinite interface crack. By contrast, for any finite crack problem, there will be no first interpenetration zone, but there will be a final zone, associated with the gradual merging of the asymptotic crack-tip field into the more complex finite-geometry field. At first sight, this seems to imply that we need to know the particular displacement field for the problem or at least some higher order terms in the asymptotic expansion in order to determine the end of the final interpenetration zone. However, each of the zones defined by Eqn (14) maps into the next in the sequence by a magnification in length scale of C^{-4} where C is given by Eqn (9). Now C^{-4} is very large—the smallest possible value is about 3.9×10^{15}—and it can easily arise that there is some n for which the maximum r defined by Eqn (14) is several orders of magnitude smaller than the crack length, whilst the minimum r defined using the *next* value $(n + 1)$ exceeds the crack length. Indeed, this condition must be satisfied if the stress field is to involve small-scale contact conditions.

Thus, under small-scale contact conditions, a good estimate of the first interpenetration point is given by the largest root of the equation

$$a_i = a\exp\left[\{(2n - \tfrac{1}{2})\pi - arg(K) + \arctan(2\varepsilon)\}/\varepsilon\right] \tag{15}$$

which is smaller than the crack length. Notice that the length scale a in this equation can be chosen arbitrarily, but that the complex stress intensity factor K has, then, to be that appropriate to the length scale chosen, to agree with the asymptotic form, Eqn (1).

5. ASYMPTOTIC FEATURES OF THE CONTACT SOLUTION

For the interface crack, the appropriate analytical treatment depends on the relative magnitudes of three distinct length scales—the process zone dimension, a_p, the contact length, a_c, and a dimension characterizing the large-scale geometry of the cracked body, a_g. For an elastic solution to be appropriate, it is necessary that $a_p \ll a_g$, since this ensures that there is a zone surrounding the process zone in which the elastic asymptotic field is dominant. Small-scale contact conditions might be defined by the relative magnitudes $a_c < a_p \ll a_g$.

A different approach to the asymptotic question can be used if instead we have $a_p \ll a_c \ll a_g$, i.e. if the process zone is small compared with the contact length, which, however, is still small compared with the large-scale geometry. In this case, the condition $a_c \ll a_g$ is sufficient to guarantee (i) that there will be a region somewhere in the range $a_c < r < a_g$ in which the elastic field has the oscillatory asymptotic form of the open solution and (ii) that the conditions in the more complicated region near the crack tip must be completely determined by the properties of this surrounding asymptotic field and hence must be characterizable in terms of two scalar parameters, e.g. the real and imaginary parts of K in Eqn (1) (with an arbitrary length scale a) or K_1, a_1 of Eqn (8). Notice that the argument to this point is identical to that of Rice [10].

However, because of the further condition $a_p \ll a_c$, the open solution no longer contains all the useful information that can be extracted from an elastic analysis, since the existence of the contact zone will change the deductions that might be made about the micromechanics of fracture from the nature of the crack-tip fields. In particular, a closed crack tip has a distinct asymptotic field of its own, characterized by a single mode II stress intensity factor and a proportional square-root singularity in contact pressure between the crack faces [3]. Thus, there can be no mode-mixity effects on the fracture criterion at a closed tip if the process zone is significantly smaller than the contact length. We conclude that, under the conditions $a_p \ll a_c \ll a_g$, the nature of the contact field provides useful information relevant to the micromechanics of fracture, but there is a one-to-one correspondence between the properties of this field and those of the asymptotics for the corresponding open solution. Furthermore, since the open asymptotics can be completely characterized in terms of a length scale a_1 and a real scalar stress-intensity factor K_1 (see Eqn (8)), it follows that all conceivable contact fields under the conditions $a_p \ll a_c \ll a_g$ can be obtained by linear scaling from a universal contact field which, in turn, could be derived from any of the existing solutions of particular problems subject to these conditions.

The relationship between the properties of the oscillatory field and the embedded contact-zone field was first investigated by Atkinson [13, 14], who, however, somewhat obscured the argument by claiming that there would be an infinite number of possible contact solutions, by analogy with the infinite sequence of solutions of Eqn (15). This apparent multiplicity of solutions is quite illusory, as indeed can be argued directly from Shield's uniqueness theorem [7]. Each of Atkinson's solutions has a one-to-one correspondence with a solution of Eqn (15) and, as in that case, we must choose the largest solution that gives a contact zone that lies within the crack dimensions, since only in this way can we ensure that the surrounding oscillatory field contains no interpenetration zones.

Atkinson's analysis is specific to the problem of the plane crack in tension and shear, though it is clear from the last paragraph of [14] that he was aware of the extension to the more general two- or even three-dimensional problem. By expressing Eqn (2.10) of [13] in terms of K, a of our Eqn (1) and using (3.13) of [13], we can show that the contact length a_c is given by

$$a_c = 4a\exp[\{(2n - \tfrac{1}{2})\pi - arg(K)\}/\varepsilon]. \tag{16}$$

where n is an integer which has to be chosen to have the maximum value that gives a value of a_c within the extent of the crack.

Alternatively, we can relate a_c to the point of first interpenetration, a_i, by eliminating K, between Eqns (15 and 16), obtaining

$$a_c = 4a_i\exp\left[-\frac{\arctan(2\varepsilon)}{\varepsilon}\right]. \tag{17}$$

In other words, the contact length will be a fixed proportion (dependent on ε) of the interpenetration length in the open solution, as long as both are sufficiently small in comparison with the large-scale geometry of the problem. Furthermore, the ratio

$$\frac{a_c}{a_i} = 4\exp\left[-\frac{\arctan(2\varepsilon)}{\varepsilon}\right] \tag{18}$$

is a very weak function of ε, varying only between 0.541 for $\varepsilon = 0$ and 0.584 for the maximum possible value, $|\varepsilon| = 0.175$.

The mode II stress-intensity factor, K_{II}, can be found similarly from [13] and is, in fact, simply the magnitude of the complex stress-intensity factor of the open field, i.e. $K_{II} = |K|$. This result may be verified by evaluating the J-integral for both the open and contact solutions, and invoking the requirement that the integral be path-independent [21]. Similarly, by extending the path of the integral to include *both* crack tips we can see that the mode II stress-intensity factor is equal and opposite at the two ends.

Other properties of the embedded contact field could be obtained in the same way, using the appropriate equations from [13].

Some estimate of the ratio a_c/a_g needed for the method to give accurate numerical results can be obtained by using the present method to estimate the contact zone size for the plane interface crack in the presence of combined shear and tension [5, 9]. In this case, the contact zone at one end of the crack increases with the ratio of tension to shear, whilst that at the other end decreases. The open solution predicts a value of K based on the crack length, L, (i.e. $a = L$ in Eqn (1)) of

$$K = \tfrac{1}{2}(T \pm iS)(1 + 2i\varepsilon)L^{1/2} \tag{19}$$

where T and S are the far-field normal and tensile stresses, respectively [10]. The asymptotic analysis therefore predicts contact zone lengths of

$$a_c = 4L\exp\left[-\left\{\frac{\pi}{2} \pm \arctan\left(\frac{S}{T}\right) + \arctan(2\varepsilon)\right\}\Big/\varepsilon\right]. \tag{20}$$

Figure 1 provides a comparison of this result with the predictions of Gautesen and Dundurs [9] for the large contact zone, with a material combination with the maximum value of $\beta = 0.5$ ($\varepsilon = 0.175$). The results indicate that the asymptotic method gives a reasonable approximation to the length of the large contact zone as long as a_c/L is sufficiently small. In fact, the asymptotic approximation consistently overestimates the contact length, the error being comparable with the ratio a_c/L. Thus, 10% accuracy is maintained for $a_c/L < 0.1$ which, for the plane interface crack and $\varepsilon = 0.175$, corresponds to a ratio of far-field shear to tension of $S/T = 2.6$. Similar results are obtained for other values of ε, the accuracy being determined by the ratio a_c/L.

We should, of course, anticipate this result, since the method presented in this section—essentially the embedding of a universal solution for the contact-zone field within a matched surrounding asymptotic field—depends not only on the existence of such a surrounding field, but also on the premise that the contact zone is sufficiently small for its influence on the large-scale field to be negligible, so that the surrounding oscillatory field is the same as that obtained with the open assumptions. This idea is exploited by Dundurs and Gautesen [15] to produce a very good closed-form approximation for the plane interface crack in combined tension/compression and shear. Noting that the effect of the shear is to produce a "large" contact zone at one crack tip, but a zone smaller even than that produced under pure tension at the other tip, they solved the problem using the traction-free boundary condition at the tip where a small zone is anticipated, which is equivalent to the assumption that the small contact zone has a negligible effect on the elastic fields except in its own immediate vicinity. The large contact zone is then correctly treated using the unilateral formulation, leading to an integral equation with only one unknown contact length, which can be solved exactly, except for the determination of a parameter from an implicit algebraic equation.

It is clear from the above discussion that the extent of the true small contact zone could then be recovered from the oscillatory asymptotic field at the end assumed open, though, of course, this zone will almost always be so small as to be of only theoretical interest.

Application to three-dimensional problems

As in other applications of the Williams' asymptotic technique, we can apply the method to general three-dimensional crack problems, provided that the geometry satisfies the additional condition that the local radius of curvature of the crack boundary be large

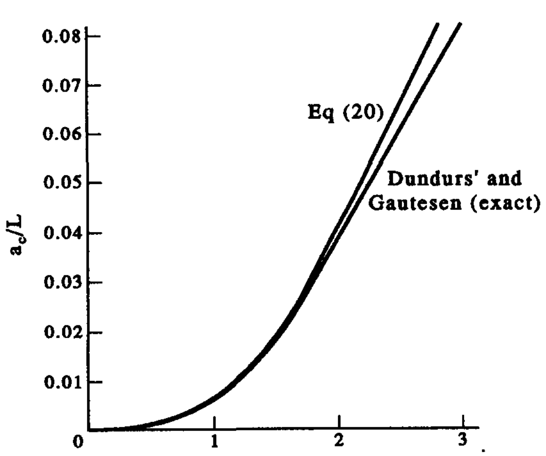

FIG. 1. A comparison of the predicted size of the larger contact zone for a plane crack subjected to combined tension (T) and shear (S).

compared with a_c, in which case a_c will be the normal distance from a local point on the boundary to the edge of the contact region.

The open solution for the penny-shaped interface crack under tension was first given by Mossakowski and Rybka [16] and Erdogan [17], and the appropriate value of K was given by Kassir and Bregman [18] as

$$K = \frac{(2c)^{1/2}}{\pi^{1/2}} \frac{\Gamma(2 + i\varepsilon)}{\Gamma(\frac{1}{2} + i\varepsilon)} T \tag{21}$$

using the crack diameter, $2c$, as scaling parameter. Substituting this value into Eqn (16) and writing $a = 2c$, where c is the crack radius, we find

$$a_c = 8c \exp\left[-\left\{ \frac{\pi}{2} + arg\left(\frac{\Gamma(2 + i\varepsilon)}{\Gamma(\frac{1}{2} + i\varepsilon)} \right) \right\} \bigg/ \varepsilon \right]. \tag{22}$$

The maximum value occurs when $\varepsilon = 0.175$ and is $a_c = 1.005 \times 10^{-4} c$. The corresponding unilateral solution was published by Keer *et al.*, who solved for the radius of the open portion of the crack to its radius. From this we can infer a value of $a_c \approx 2 \times 10^{-4} c$, but the discrepancy is probably an indication of difficulties in their numerical treatment of the integral equation, and the insensitivity of the technique to the ratio a_c/c. Thus, this difference should not be interpreted as a deviation from the dominant term in the asymptotic field, and we note that the result clearly satisfies the requirement that $a_c \ll a_g$.

Perhaps a more interesting application is to the penny-shaped interface crack in combined shear and tension, for which the open solution is given by Willis [19], but for which the contact solution has not so far been solved. The shear component, S, of the traction can be taken to be in the x-direction without loss of generality. We then find that the tensile loading, T, produces a uniform oscillatory field around the crack edge, as defined by Eqn (21), whereas the shear loading produces an oscillatory field varying with $\cos(\theta)$ and a square-root (non-oscillatory) mode III field varying with $\sin(\theta)$, where (r, θ) are polar coordinates positioned at the crack centre.

The oscillatory field can be defined in terms of a function $K(\theta)$ where

$$\sigma_{zz} + i\sigma_{rz} = K(\theta)(r - c)^{-1/2} \left(\frac{r - c}{2c} \right)^{i\varepsilon} \tag{23}$$

and

$$K(\theta) = \frac{(2c)^{1/2}}{\pi^{1/2}} \frac{\Gamma(2 + i\varepsilon)}{\Gamma(\frac{1}{2} + i\varepsilon)} \left\{ T + \frac{2iS\cos\theta}{\left[\dfrac{(1 - \beta^2)\pi\varepsilon(1 + \varepsilon^2)}{4\beta\gamma} + 1 \right]} \right\}. \tag{24}$$

The mode III field is defined by the usual stress-intensity factor

$$K_{\mathrm{III}}(\theta) = \underset{r \to c}{Lt}\, (r - c)^{1/2} \sigma_{\theta z} = -\frac{2(2a)^{1/2} \varepsilon(1 + \varepsilon^2) S \sin\theta}{\left[\pi\varepsilon(1 + \varepsilon^2) + \dfrac{4\beta\gamma}{(1 - \beta^2)} \right]} \tag{25}$$

where, in all these equations, the diameter of the crack, $2c$, is taken as the length scale.

The extent of the contact zone a_c, is determined only by the oscillatory component in the field and is therefore given as a function of θ by

$$a_c(\theta) = 8c \exp\left[-\left\{ \frac{\pi}{2} + arg\left[\frac{\Gamma(2 + i\varepsilon)}{\Gamma(\frac{1}{2} + i\varepsilon)} \left(T + \frac{2iS\cos\theta}{\left[\dfrac{(1 - \beta^2)\pi\varepsilon(1 + \varepsilon^2)}{4\beta\gamma} + 1 \right]} \right) \right] \right\} \bigg/ \varepsilon \right]. \tag{26}$$

This result is to be interpreted in the sense that the equation defining the inner central separation region is

$$r = c - a_c(\theta). \tag{27}$$

The maximum value of a_c occurs at $\theta = 0$ and will be less than 10% of the diameter ($2c$) for $\varepsilon = 0.175$ provided that $S/T < 2.9$ (when $\gamma = \frac{1}{2}$) or 3.7 (when $\gamma = \frac{1}{4}$). For larger ratios of

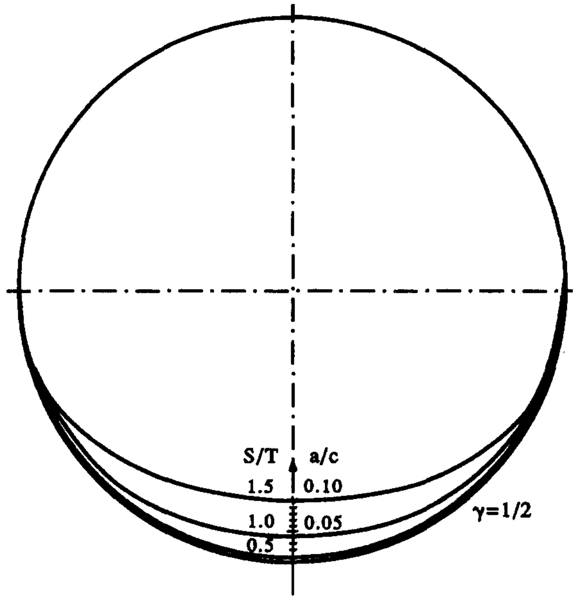

FIG. 2. The shape of the predicted contact zone for a penny-shaped interface crack under combined tension (T) and shear (S), $\gamma = \frac{1}{2}$.

shear to tension, it would be necessary to solve the full three-dimensional problem with unilateral boundary conditions. The variation of size of the contact zone with an angular position around the edge of the contact is shown in Fig. 2, for various values of S/T, and $\gamma = 1/2$.

6. SITUATIONS WHERE THE MISMATCH PARAMETER IS SMALL

The mismatch in material properties at an interface produces qualitative differences in the elastic solution, most notably the oscillatory nature of the surrounding field and the generation of contact zones. However, if the mismatch parameter ε is allowed to tend to zero, the stress and displacement fields will tend continuously towards the "uncoupled" limit, characterized by non-oscillatory fields and the absence of contact zones (except in the presence of far-field compressive tractions). In particular, this transition will involve a continuous reduction in the angle of the logarithmic spiral defined by $\theta = arg(\sigma_{yy}(r) + i\sigma_{xy}(r))$ on any radial line, the effective extension to infinity of the wavelength of the oscillations at any given radius, characterized by the constant C of Eqn (9) tending to zero, and the shrinkage to zero of the characteristic lengths a_i, a_c defined by Eqns (15 and 16).

Now all these quantities are, in a sense, near their limits even in the most extreme case $\varepsilon = 0.175$. For example, the predicted contact zones are often extremely small and the maximum value of the constant C is only 0.000126. Furthermore, in many practical cases, ε will be significantly lower than the maximum and the stresses at all points outside the process zone may be almost indistinguishable from those that would be obtained for a material pair with $\varepsilon = 0$. In such cases, there is a strong argument for developing an approximate solution based on setting ε to zero, leading to simple square-root singularities, conventional stress-intensity factors and permitting classical fracture mechanics arguments to be used [20].

However, as in the above discussion of small-scale contact conditions, it is dangerous to assume that all dissimilar material problems can be adequately characterized by the solution of a problem with $\varepsilon = 0$. A single example which will serve to illustrate the dangers of this oversimplification is provided by the plane interface crack with far-field shear and compression. If ε is assumed to be zero, the crack will close everywhere and equal mode II stress-intensity factors will be developed at each end. By contrast, even a very small amount of mismatch between the material properties (described by a small value of ε) will require there to be a zone of separation very near to one of the crack tips. This is almost certain to result in a tendency for preferential fracture at the nearly open end and is therefore a phenomenon that must be adequately addressed in a micromechanical analysis of the fracture process.

7. CONCLUSIONS

We have collated and compared the fully open and unilateral solutions for both plane and circular interface cracks, under far-field tension and shear.

In the context of linear elasticity, the unilateral contact formulation provides the only correct solution for an interface crack when $\beta \neq 0$. However, the simpler "open" solution can be used in those cases where the predicted contact zone is smaller than the process zone in which real materials would be expected to exhibit non-linear effects.

In the present paper, we have extended the range in which the open solution can be used by showing that a good approximation to the unilateral (contact) solution can be constructed by embedding an appropriate "contact" field within the asymptotic crack-tip field associated with the open solution. In particular, the length of the contact zone can be predicted from the distance between the point of first interpenetration and the crack tip in the open solution. This approximation has been tested against published solutions to various interface crack problems and is also used to estimate the extent of the contact region for the penny-shaped crack loaded in shear and tension—a problem that has so far not been solved in the unilateral formulation.

The burden of the paper is to highlight the following procedure for the treatment of interface cracks:

1. The crack should first be analyzed assuming that it is open, and the interpenetration distance found.
2. If the interpenetration distance at all crack tips is not small compared with the crack length or other dimensions in the problem geometry, this normally implies that the contact zone is large, and a full unilateral solution is demanded.
3. If the predicted interpenetration distances are so small that they are expected to be contained within the process zone, the open solution contains all the information that can be relevant in determining fracture using a linear elastic fracture mechanics criterion.
4. However, if the interpenetration distances are sufficiently small in the sense of (2) above (e.g. $a_i/L < 0.1$ for 10% accuracy), but are significantly larger than the expected process zone dimensions, a sufficiently accurate unilateral solution can be constructed by embedding an appropriate contact field within the asymptotic field associated with the open solution and fracture mechanics arguments can then be applied to the local properties of this embedded field. The contact length for the embedded field can be obtained from Eqn (16) and the mode II stress-intensity factor is equal to the magnitude of the complex stress-intensity factor for the open solution. Other more detailed properties of the contact field can be obtained by substituting these results into appropriate equations in [13].

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REFERENCES

1. F. ERDOGAN, Stress distribution in bonded dissimilar materials with cracks. *ASME J. Appl. Mech.* **32**, 403–410 (1965).
2. A. H. ENGLAND, A crack between dissimilar media. *ASME J. Appl. Mech.* **32**, 400–402 (1965).
3. M. COMNINOU, The interface crack. *ASME J. Appl. Mech.* **44**, 631–636 (1977).
4. M. COMNINOU, The interface crack in a shear field. *ASME J. Appl. Mech.* **45**, 287–290 (1978).
5. M. COMNINOU and D. SCHMEUSER, The interface crack in a combined tension–compression and shear field. *ASME J. Appl. Mech.* **46**, 345–348 (1979).

6. L. M. KEER, S. H. CHEN and M. COMNINOU, The interface penny-shaped crack reconsidered. *Int. J. Engng Sci.* **16**, 765–772 (1978).
7. R. T. SHIELD, Uniqueness for elastic crack and punch problems. *ASME J. Appl. Mech.* **49**, 516–518 (1982).
8. A. K. GAUTESEN and J. DUNDURS, The interface crack in a tension field. *ASME J. Appl. Mech.* **54**, 93–98 (1987).
9. A. K. GAUTESEN and J. DUNDURS, The interface crack under combined loading, *ASME J. Appl. Mech.* **55**, 580–586 (1988).
10. J. R. RICE, Elastic fracture mechanics concepts for interfacial cracks. *ASME J. Appl. Mech.* **55**, 98–103 (1988).
11. M. L. WILLIAMS, Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *ASME J. Appl. Mech.* **19**, 526–528 (1952).
12. J. DUNDURS, Discussion on "Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading". *ASME J. Appl. Mech.* **36**, 650–652 (1969).
13. C. ATKINSON, The interface crack with a contact zone (an analytical treatment). *Int. J. Fracture* **18**, 161–177 (1982).
14. C. ATKINSON, The interface crack with a contact zone—2. (The crack of finite length). *Int. J. Fracture* **19**, 131–138 (1982).
15. J. DUNDURS and A. K. GAUTESEN, An opportunistic analysis of the interface crack, *Int. J. Fracture* **36**, 151–159 (1988).
16. W. I. MOSSAKOWSKI and M. T. RYBKA, Generalization of the Griffith–Sneddon criterion for the case of a non-homogeneous body. *J. Appl. Math.* **28**, 1061–1069 (1964).
17. F. ERDOGAN, Stress distribution on bonded dissimilar materials containing circular or ring-shaped cavities. *ASME J. Appl. Mech.* **32**, 829–836 (1965).
18. M. K. KASSIR and A. M. BREGMAN, The stress-intensity factor for a penny-shaped crack between two dissimilar materials. *ASME J. Appl. Mech.* **39**, 308–310 (1972).
19. J. R. WILLIS, The penny-shaped crack on an interface. *Q. J. Mech. Appl. Math.* **25**, 367–385 (1972).
20. Z. SUO and J. W. HUTCHINSON, Steady-state cracking in brittle substrate beneath adherent films, Harvard Univ. Report No. Mech—132.
21. N. ARAVAS and S. M. SHARMA, An elastoplastic analysis of the interface crack with contact zones. *J. Mech. Phys. Solids* **39**, 311–344 (1991).