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SOME MIXED BOUNDARY VALUE PROBLEMS FOR THE SEMI-INFINITE ELASTIC SOLID SUBJECTED TO TANGENTIAL SURFACE TRACTIONS

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Summary—A method is developed for representing the tangential displacements and tractions at the surface of the semi-infinite solid in terms of potential functions. In this form, a mathematical analogy is revealed between corresponding mixed boundary value problems involving tangential and normal surface displacements respectively. This analogy enables a general solution to be obtained to the problem in which the surface tangential displacements are specified axisymmetric functions inside the circle $a \geqslant r \geqslant 0$ and the tangential surface traction is zero outside this circle. The method can also be used for certain non-axisymmetric problems, but it fails if the indentation analogue has a stress singularity at the boundary of the stressed area.

NOTATION

r, θ, z	cylindrical polar co-ordinates
x, y, z	rectangular Cartesian co-ordinates
a	radius of a circular area on the surface of the semi-infinite solid
E	Young's modulus
e	surface dilatation
P	concentrated force acting at the surface
$p_{rr}, p_{\theta z}$, etc.	stress components in double suffix notation
u_r, u_z, u_θ	components of surface displacement
∇^2	$\equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$
ν	Poisson's ratio
ϕ, ψ	stress functions
ξ, η	displacement functions
ζ	$= \xi + i\eta$
ω	surface rotation

1. INTRODUCTION

The problems to be considered in this paper are the tangential analogues of the axisymmetric indentation problem associated with the name of Boussinesq and of certain related non-axisymmetric problems. We shall find the distribution of tangential traction within the circle $a \geqslant r \geqslant 0$ on the surface of the semi-infinite elastic solid which is necessary to produce prescribed tangential surface displacements within this circle, the rest of the surface being stress free. The semi-infinite solid is considered to occupy the space $z > 0$ in the cylindrical polar co-ordinate system r, θ, z.

With this notation, a formal statement of the above boundary conditions is:

normal surface traction: $p_{zz} = 0, \quad z = 0, \quad \text{all } r, \theta,$

tangential surface tractions: $p_{rz} = p_{\theta z} = 0, \quad z = 0, \quad r > a,$

tangential surface displacements:
u_r, u_θ prescribed functions of $r, \theta, \quad z = 0, \quad a \geqslant r \geqslant 0.$

Various particular solutions of this problem are known, notably those involving symmetric torsion of the semi-infinite solid about the z axis. Weinstein[1] derived the solution for a solid containing a penny-shaped crack in torsion about its axis of symmetry and Payne[2] showed that this and other solutions can be obtained from existing results by virtue of a mathematical analogy between axisymmetric torsion and axisymmetric indentation.

In this paper, we shall demonstrate the existence of another, more general, analogy between these classes of problems, which is not restricted either to torsional or to axisymmetric systems.

2. REPRESENTATION OF TANGENTIAL SURFACE DISPLACEMENT AND TRACTION BY POTENTIAL FUNCTIONS

The tangential displacement at the surface of the semi-infinite solid is a vector with two components (e.g. u_r, u_θ). It is convenient to define this vector in terms of the gradient of a potential function in order to facilitate the transformation of co-ordinate systems. Following the methods of two-dimensional elasticity, we find that the most general definition of this type requires two independent scalar functions ξ, η, in terms of which

$$u_x = \frac{\partial \xi}{\partial x} - \frac{\partial \eta}{\partial y}; \quad u_y = \frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}. \tag{1}$$

It is easily verified that if n, t are two orthogonal directions, inclined to x, y respectively at some angle θ, we have

$$u_n = \frac{\partial \xi}{\partial n} - \frac{\partial \eta}{\partial t}; \quad u_t = \frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial n} \tag{2}$$

(see, for example, Dugdale,[3] Section 1:6).

A more compact statement of equation (2) is

$$u_n + iu_t = \frac{\partial \zeta}{\partial n} + i\frac{\partial \zeta}{\partial t}, \tag{3}$$

where

$$\zeta = \xi + i\eta. \tag{4}$$

The tangential surface traction (components $p_{rz}, p_{\theta z}$) also lends itself to this form of representation. We shall use the two stress functions ϕ, ψ, where

$$p_{nz} = \frac{\partial \phi}{\partial n} - \frac{\partial \psi}{\partial t}; \quad p_{tz} = \frac{\partial \phi}{\partial t} + \frac{\partial \psi}{\partial n}. \tag{5}$$

We define the surface dilatation, e, by the equation

$$e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}, \tag{6}$$

$$= \nabla^2 \xi \tag{7}$$

from equation (1).

Similarly, we define the surface rotation, ω, by

$$2\omega = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}, \tag{8}$$

$$= \nabla^2 \eta. \tag{9}$$

Equations (7) and (9) provide a convenient means of finding ξ, η, for a given displacement field.

3. THE CONCENTRATED TANGENTIAL FORCE

If a concentrated tangential force, P, acts at the origin on the boundary of the semi-infinite solid $z > 0$, in the direction of the x axis, the surface displacements will be

$$u_x = \frac{P(1+\nu)}{\pi E r}\left(1-\nu+\frac{\nu x^2}{r^2}\right), \tag{10}$$

$$u_y = \frac{P(1+\nu)\,\nu x y}{\pi E r^3}, \tag{11}$$

$$u_z = \frac{P(1+\nu)(1-2\nu)\,x}{2\pi E r^2} \tag{12}$$

(see Love[4], Art. 166), where E, ν, are Young's modulus and Poisson's ratio respectively, for the material.

From equations (7), (9), (10) and (11) we have

$$\nabla^2 \xi = -\frac{P(1-\nu^2)\,x}{\pi E r^3} \tag{13}$$

and

$$\nabla^2 \eta = \frac{P(1+\nu)\,y}{\pi E r^3} \tag{14}$$

giving

$$\zeta = \frac{P}{\pi E}\left((1-\nu^2)\frac{x}{r} - i(1+\nu)\frac{y}{r}\right). \tag{15}$$

4. DISPLACEMENT POTENTIAL DUE TO A DISTRIBUTION OF TANGENTIAL TRACTION

We suppose that the semi-infinite solid is subjected to tangential tractions defined by equation (5), where ϕ, ψ are single-valued functions. This condition will ensure that there are no concentrated forces acting at the surface.

We further suppose that there is some finite closed region outside which the surface is stress free. In this unstressed region we shall take ϕ, ψ to be zero. These restrictions require that ϕ, ψ should approach zero at the boundary of the stressed region.

Consider the displacement potential at an arbitrary point (A) due to the traction on an element of surface at another point (B). We choose A as the origin of a polar co-ordinate system, relative to which B has co-ordinates (r, θ). The area of the surface element is $(r\,\mathrm{d}\theta\,\mathrm{d}r)$.

From equation (15) we have

$$\mathrm{d}\zeta = \left(-\frac{p_{rz}}{\pi E}(1-\nu^2) - \frac{i p_{\theta z}(1+\nu)}{\pi E}\right) r\,\mathrm{d}\theta\,\mathrm{d}r \tag{16}$$

and hence

$$\mathrm{d}\xi = -\frac{1-\nu^2}{\pi E}\left(r\frac{\partial \phi}{\partial r} - \frac{\partial \psi}{\partial \theta}\right)\mathrm{d}\theta\,\mathrm{d}r; \tag{17}$$

$$\mathrm{d}\eta = -\frac{1+\nu}{\pi E}\left(r\frac{\partial \psi}{\partial r} + \frac{\partial \phi}{\partial \theta}\right)\mathrm{d}\theta\,\mathrm{d}r. \tag{18}$$

We now integrate over the surface, noting that

$$\int_0^{2\pi} \frac{\partial \phi}{\partial \theta} \mathrm{d}\theta = \int_0^{2\pi} \frac{\partial \psi}{\partial \theta} \mathrm{d}\theta = 0, \tag{19}$$

since ϕ, ψ have been assumed to be single valued. Thus,

$$\xi_A = -\frac{1-\nu^2}{\pi E} \int_0^{2\pi} \int_0^{\infty} r \frac{\partial \phi}{\partial r} \mathrm{d}r \, \mathrm{d}\theta, \tag{20}$$

$$= \int_0^{2\pi} \left(-\frac{(1-\nu^2) r\phi}{\pi E} \right)_{r=0}^{r=\infty} \mathrm{d}\theta + \int_0^{2\pi} \int_0^{\infty} \frac{(1-\nu^2) \phi \, \mathrm{d}r \, \mathrm{d}\theta}{\pi E}. \tag{21}$$

The conditions which we have imposed on the function ϕ ensure that the integrand of the first term is zero for all θ, provided that ϕ is bounded at A.

Hence,

$$\xi_A = \int_0^{2\pi} \int_0^{\infty} \frac{(1-\nu^2) \phi \, \mathrm{d}r \, \mathrm{d}\theta}{\pi E}. \tag{22}$$

By a similar argument, we also have

$$\eta_A = \int_0^{2\pi} \int_0^{\infty} \frac{(1+\nu) \psi \, \mathrm{d}r \, \mathrm{d}\theta}{\pi E}. \tag{23}$$

5. ANALOGY WITH THE NORMAL INDENTATION PROBLEM

Equations (22) and (23) reveal an important analogy between tangential and normal loading of the semi-infinite solid. The normal surface displacement (u_z) of the semi-infinite solid at the origin due to a distribution of normal pressure $p\ (r, \theta)$ is

$$u_z = \int_0^{2\pi} \int_0^{\infty} \frac{(1-r^2) p \, \mathrm{d}\theta \, \mathrm{d}r}{\pi E} \tag{24}$$

(see Timoskenko and Goodier,[5] Art. 139). which has the same form as equations (22) and (23). Note that in all these results the choice of origin of co-ordinates is arbitrary and hence equations (22)–(24) apply throughout the surface plane, provided only that ϕ, ψ, p are suitably defined relative to the appropriate origin.

Various methods[6–8] have been developed for finding the pressure distribution p, necessary to produce a prescribed displacement u_z, over a closed (usually circular) region and we can use these methods directly to find corresponding solutions to the problem defined in Section 1.

As an illustrative example, it is well known that the pressure distribution,

$$\left.\begin{aligned} P &= c\sqrt{(a^2-r^2)}, \quad a \geqslant r \geqslant 0, \\ &= 0, \quad r > a, \end{aligned}\right\} \tag{25}$$

produces a paraboloidal depression defined by

$$u_z = \frac{\pi c(1-\nu^2)(2a^2-r^2)}{4E}, \quad a \geqslant r \geqslant 0 \tag{26}$$

(Timoshenko and Goodier[5], Art. 140).

By virtue of the analogy between equations (22) and (24), it follows that a stress potential

$$\left.\begin{aligned} \phi &= c\sqrt{(a^2-r^2)}, \quad a \geqslant r \geqslant 0, \\ &= 0, \quad r > a, \end{aligned}\right\} \tag{27}$$

corresponding to a tangential stress distribution

$$\left.\begin{aligned} p_{rz} &= -\frac{cr}{\sqrt{(a^2-r^2)}} \quad a \geqslant r \geqslant 0, \\ &= 0 \quad r > a, \\ p_{\theta z} &= 0 \quad \text{all } r, \end{aligned}\right\} \tag{28}$$

from equations (2), (27), will produce a displacement potential

$$\xi = \frac{\pi c(1-\nu^2)(2a^2-r^2)}{4E}, \quad a \geqslant r \geqslant 0, \tag{29}$$

and hence displacements

$$\left.\begin{aligned} u_r &= -\frac{\pi c(1-\nu^2)\,r}{2E} \quad a \geqslant r \geqslant 0 \\ u_\theta &= 0 \quad \text{all } r. \end{aligned}\right\} \tag{30}$$

This represents a uniform axisymmetric dilatation of the surface within the circle $a \geqslant r \geqslant 0$. The dilatation

$$e = -\frac{\pi c(1-\nu^2)}{2E}, \tag{31}$$

from equations (7) and (29).

The same solution used in conjunction with equation (23) shows that a stress distribution

$$\left.\begin{aligned} p_{\theta z} &= -\frac{cr}{\sqrt{(a^2-r^2)}}, \quad a \geqslant r \geqslant 0, \\ &= 0, \quad r > a; \\ p_{rz} &= 0, \quad \text{all } r, \end{aligned}\right\} \tag{32}$$

produces a displacement

$$\left.\begin{aligned} u_r &= 0, \\ u_\theta &= -\frac{\pi c(1+\nu)\,r}{2E}, \quad a \geqslant r \geqslant 0. \end{aligned}\right\} \tag{33}$$

This solution represents a constant rotation of the circle $a \geqslant r \geqslant 0$ about the origin and corresponds to the case in which a rigid circular cylinder is joined to the semi-infinite solid over the circle and is then twisted about its axis. The rotation

$$\omega = -\frac{\pi c(1+\nu)}{2E}. \tag{34}$$

6. THE GENERAL AXISYMMETRIC SOLUTION

The general solution of the axisymmetric tangential displacement problem presents no new difficulties. We first express the required displacement within the circle $a \geqslant r \geqslant 0$ in terms of the displacement potentials ξ, η, using equations (7) and (9). We then solve separately for the distribution of the stress functions ϕ, ψ, respectively, necessary to produce these potentials, making use of the analogies between equations (22) and (24) and one of the known methods for solving equation (24). Since we require the contact pressure p only in equation (24), the methods of Green[6] and Segedin[7] prove to be the most straightforward. Finally, we find the corresponding tangential stresses in the region $a \geqslant r \geqslant 0$ by substituting for ϕ, ψ, in equation (2).

The solution of the corresponding indentation problem has been very extensively discussed in the cited references and elsewhere and will not be considered here. However, we note that the complete solution outlined above compares favourably in mathematical simplicity with other methods (for example, the use of Westmann's[9] solution for the elastic half-space in shear).

7. RESTRICTIONS ON THE STRESS POTENTIAL AND SINGULARITIES

In Section 4, we laid down certain conditions on the values of ϕ, ψ, some of which can now be relaxed.

(a) *Step changes in stress potential*

Suppose there exists a step change ϕ_0 in the stress function ϕ across some line S. We could regard such a step as the limiting case of a ramp of slope ϕ_0/δ and width δ. From equation (2), the stress normal to S will be

$$p_{nz} = \phi_0/\delta \tag{35}$$

and hence the ramp corresponds to a force ϕ_0 per unit length of line, which remains constant as we approach a step by allowing δ to tend to zero. The force acts normal to the line of the step in the direction of increasing ϕ.

Similarly, a step change, ψ_0, in the stress function ψ corresponds to a force ψ_0 per unit length, tangential to the line of the step.

(b) "*Point force*" *singularities*

Analogues of the point force normal to the surface can be found by supposing that ϕ, ψ are zero everywhere on the surface except over a small circle of radius δ with its centre at the origin.

If $\phi = \phi_0$ over this circle, there will be a radial force ϕ_0 per unit length acting around the circumference towards the origin. If we allow δ to approach zero whilst $(\pi\delta^2\phi_0)$ remains constant and equal to Φ_0, we generate a singularity which we can describe as a centre of dilatation by analogy with the corresponding singularity in two-dimensional elasticity. The displacement potential due to this singularity is

$$\xi = \frac{\Phi_0(1-\nu^2)}{\pi E r}. \tag{36}$$

The corresponding singularity in ψ is the "centre of rotation", ψ_0, which represents a concentrated moment acting about an axis normal to the surface and which produces the displacement potential

$$\eta = \frac{\Psi_0(1+\nu)}{\pi E r}. \tag{37}$$

The formal analogy between these singularities and the normal displacement

$$u_z = \frac{P(1-\nu^2)}{\pi E r}, \tag{38}$$

due to a normal point force P, provides an alternative method of deriving the results of Section 4.

(c) *Behaviour of* ϕ, ψ *at large values of* r

In Section 4, we assumed that there is some finite closed region outside which ϕ, ψ are identically zero, but it is clear from equation (21) that the argument would still be valid provided that $r\phi, r\psi \to 0$ as $r \to \infty$. If this condition is not satisfied, it will usually be possible to meet it by "subtracting out" a uniform state of stress, as in the solution of crack problems.

8. NON-AXISYMMETRIC PROBLEMS

Although every normal identation solution has a tangential displacement analogue and vice versa, not all these solutions represent physically possible situations. An axisymmetric example is provided by the flat-ended punch problem, the tangential analogue of which has an unacceptable singularity, involving an infinite concentrated force, at the outer radius $r = a$. This is not a serious difficulty in the axisymmetric case since all physically possible axisymmetric tangential displacement problems prove to have well-behaved indentation analogues.

However, when we try to extend the method to non-axisymmetric systems, using the results of Green,[6] we find that this is no longer the case. The simplest example is provided by the tilted punch "solution". A normal pressure distribution

$$\left.\begin{aligned} p &= \frac{2cEx}{\pi(1-\nu^2)\sqrt{(a^2-r^2)}}, \quad a \geqslant r \geqslant 0, \\ &= 0, \quad r > a, \end{aligned}\right\} \tag{39}$$

is known to produce a normal displacement

$$u_z = cx, \quad a \geqslant r \geqslant 0. \tag{40}$$

If we write ξ for u_z and ϕ for p, following the analogy between equations (22) and (24) and substitute in equation (1) we obtain

$$u_x = c, \quad u_y = 0, \quad a \geqslant r \geqslant 0, \tag{41}$$

which corresponds to the "rigid body" displacement of the circle $a \geqslant r \geqslant 0$ in the x direction, relative to the extremities of the solid.

However, the corresponding stress distribution from equations (5) and (39) has a physically unacceptable singularity at $r = a$.

The actual stress distribution needed to produce the displacements defined by equation (41) can be shown by integration of the point tangential force solution (equations (10) and (11)) to be

$$\left.\begin{aligned} p_{xz} &= \frac{2cE}{\pi(1+\nu)(2-\nu)\sqrt{(a^2-r^2)}}, \quad a \geqslant r \geqslant 0, \\ &= 0, \quad r > a, \\ p_{yz} &= 0, \quad \text{all } r \end{aligned}\right\} \tag{42}$$

which corresponds to the stress potentials

$$\left.\begin{aligned} \phi &= -\frac{2cE\sqrt{(a^2-r^2)}x}{\pi(1+\nu)(2-\nu)r^2}, \quad a \geqslant r \geqslant 0, \\ \psi &= +\frac{2cE\sqrt{(a^2-r^2)}y}{\pi(1+\nu)(2-\nu)r^2}, \quad a \geqslant r \geqslant 0, \\ \phi &= \psi = 0, \quad r > a. \end{aligned}\right\} \tag{43}$$

These potentials in turn have no physically possible indentation analogue because of the singularity at $r = 0$, which is self-cancelling in the tangential displacement problem. The results of equation (42) have been derived using a different method by Mindlin.[10]

9. CONCLUSION

The analogy between normal and tangential displacement of the surface of the semi-infinite solid, developed in Section 4, enables us to give a general solution to the mixed boundary value problem in which the tangential surface displacement is specified as an axisymmetric function inside the circle $a \geqslant r \geqslant 0$, and the tangential surface traction is zero outside this circle. The method finds its primary application in problems for which only surface values of stress and displacement are required, since these can be found directly without analysing the state of stress within the solid.

A similar approach can be used for corresponding non-axisymmetric problems, but it fails if the indentation analogue has a stress singularity at the boundary of the contact area.

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